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# The fuzzy cognitive pairwise comparisons for ranking and grade clustering to build a recommender system: An application of smartphone recommendation

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ABSTRACT

In a competitive high-end product market, many enterprises offer a variety of products to compete the market shares in different segments. Due to rich information of plenty of competitive product alternatives, consumers face the challenges to compare and choose the most suitable products. Whilst a product comprises different tangible and intangible features, consumers tend to buy the features rather than a product itself. A successful product has most features meeting the consumer needs. Perception values of product features from consumers are complex to be measured and predicted. To reduce information overload for searching their preferred products, this paper proposes the Fuzzy Cognitive Pairwise Comparison for Ranking and Grading Clustering (FCPC-RGC) to build a recommender system. The fuzzy number enables rating flexibility for the users to handle rating uncertainty. The Fuzzy Cognitive Pairwise Comparison (FCPC) is used to evaluate consumer preferences for multiple features of a product by pairwise comparison ratings. The Fuzzy Grade Clustering (FGC) is used to group the product alternatives into different consumer preference grades. To verify the validity and applicability of FCPC-RGC, a smartphone recommender system using the proposal approach is demonstrated how the system is able to help the consumers to recommend the suitable products according to the customers' individual preference.

## 1. Introduction

With rapid product launches from many enterprises to compete the shares in different market segments, consumers may face the challenges to explore and compare to find the most suitable products from the rich products information without sufficient market knowledge. The recommender systems perform the essential information retrieval tasks to recommend the appropriate items to the consumers. Review of recommender systems can be found in (Adomavicius and Tuzhilin, 2005; Bobadilla et al., 2013; Manouselis and Costopoulou, 2007; Burke, 2002).

There are three major categories of recommender systems: collaborative filtering (Goldberg et al., 1992; Herlocker et al., 2004; Shi et al., 2014), content-based filtering (Lops et al., 2011; Pazzani and Billsus, 2007), and hybrid approaches (de Campos et al., 2010; Salter and Antonopoulos, 2006; Pazzani, 1999). Collaborative filtering relies on the ratings from the other users to form patterns to predict a user's rating preference, whilst content-based filtering relies on items information with user ratings to make prediction, and hybrid filtering is the combination among these two methods and/or the other methods in different ways. Both collaborative filtering and content-based filtering have the drawbacks for the 'ramp-up' problems of new users and new items (Burke, 2002). In this paper, the proposed hybrid approach does not rely on the other users' rating history but the individual users' rating preferences to make recommendation results.

Since the high-end products are with short product life cycles, the classical approaches exploring historical data such as obsolete rating scores and products content may not be suitable to adaptively recommend the latest products. For the trending products, the recommender system should mainly consider the most recent data. Since a high-end product has many attributes (or features) perceived differently by diverse users, evaluating users' preferences with respect to multiple attributes is a complicated process. Multi-criteria decision making approaches have been used to evaluate users' preferences with respect to multiple attributes, for example, (Manouselis and Costopoulou, 2007; Adomavicius and YoungOk, 2007; Lakiotaki and Matsatsinis, 2011) . In this study, the Fuzzy Cognitive Pairwise Comparison (FCPC) (Yuen, 2009, 2014a) is used to evaluate the users' preferences for multi-criteria ranking and grade clustering in a product recommendation system. FCPC is the extension of the CPC (Yuen, 2009, 2012a, 2014b) with fuzzy sets.

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The FCPC (Yuen, 2009, 2014a, 2012a, 2014b) is used to elicit the users' preferences by comparing a series of the preferences of paired objects with fuzzy rating variables. An example of FCPC interface is shown in Fig. 2 in Section 2.4. With a reference object for paired comparisons, users' preference should be captured with better granularity. Conventionally, the direct rating with Likert scales and without the reference objects is the popular method for evaluation. Likert scales can be represented in fuzzy numbers. For example, (Cao and Li, 2007) demonstrated a direct rating method using fuzzy numbers to evaluate the consumer electronic products. The direct rating may not work better than the pairwise comparisons, as preference in nominal scale represented by direct rating scores may be relatively too subjective to be defined. Regarding a pairwise comparison approach, (Rokach and Kisilevich, 2012) demonstrated an approach using AHP's paired ratio scale (Saaty, 1977, 1980, 2005) for a recommender system; (Liu and Shih, 2005) integrated AHP, K-means clustering, and association rule mining in terms of recency, frequency, monetary (RFM) features for the product recommendations; (Işıklar and Büyüközkan, 2007) applied AHP and TOPSIS to evaluate mobile phones. However, there are a lot of debates for the inappropriateness of the AHP (Belton and Gear, 1983; Dyer, 1990; Forman, 1993; Belton and Goodwin, 1996; Forman and Gass, 2001; Gass, 2005; Whitaker, 2007; Bernasconi et al., 2010, 2011; Koczkodaj, 1993; Koczkodaj and Szwarc, 2014; Koczkodaj et al., 2016).

The FCPC is the core component of the Cognitive Network Process (CNP) (Yuen, 2009, 2014a, 2012a, 2014b) which is the ideal alternative of the Analytic Hierarchy Process (AHP) (Saaty, 1977, 1980, 2005) potentially producing wrong applications. The core idea of AHP relies on the paired ratio scale. The basic numerical definition of the paired ratio scale does not always appropriately represent the human intuitive judgement of paired difference, and thus CNP uses paired interval scale to replace paired ratio scale. The inappropriate definition of paired ratio scale for AHP follows the inappropriate Fuzzy AHP (FAHP), as the FAHP applies fuzzy number to the paired ratio scale. Extent Analysis Method (EAM) (Chang, 1996), the most popular approach for the FAHP, has been progressively applied in various areas, but relatively recent research (Wang et al., 2008; Wang and Chin, 2008; Yuen, 2012b) has showed the fallacy of the EAM.

The proposed Fuzzy Grade Clustering (FGC) method is used to efficiently cluster the data into ordinal grades such as low, medium, and high. The *k*-means clustering method (MacQueen, 1967) is widely used to cluster the data into different nominal groups. The *k*-means method, however, produces the local optimal clusters due to random initial centers. Inappropriate choices of initial centers lead to poor results by k-means. To address this issue, the k-mean process can be repeated many times to achieve the best cluster results, but a lot of computational workloads are required. In addition, the k-means method cannot adopt the fuzzy data and the factor weights are not considered. Fuzzy c-means (Dunn and Fuzzy, 1973)cannot deal with fuzzy input data, although it has the name "fuzzy", and in fact it is still a probabilistic method in the calculation process. The proposed FGC offsets the above shortages to grade the fuzzy weighted data in ordinal level to provide the better recommendation results.

The rest of this article is organized as follows. Section 2 proposes the novel Fuzzy Cognitive Pairwise Comparison for Ranking and Grade Clustering (FCPC-RGC) for recommender systems. Section 3 presents the validity and applicability of the proposed hybrid method. Section 4 concludes the notion of the FCPC-RGC.

## 2. Fuzzy cognitive pairwise comparison for ranking and grade clustering

The framework of the proposed Fuzzy Cognitive Pairwise Comparison for Ranking and Grade Clustering (FCPC-RGC) is illustrated in Fig. 1. A product of mixed features can be organized as a feature specification. According to the feature specification served as data schema, the items data are fetched from various sources such as mobile retailers, company engineers and customers. As some feature values are with uncertainty,

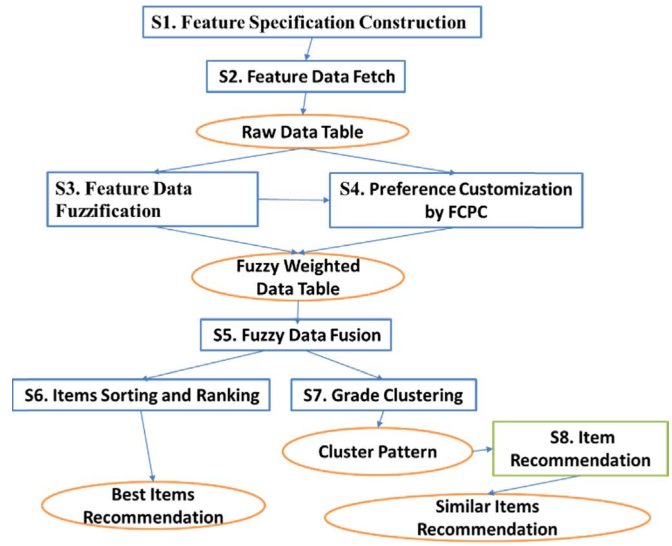


**Fig. 1.** Framework of FCPC-RGC.

their crisp values could be converted to fuzzy numbers by some fuzzification functions. Since different consumers perceive the feature values differently, the customer preference customization by Fuzzy Cognitive Pairwise Comparison (FCPC) is to elicit the current consumer preference on the feature weights and nominal feature values.

From steps 1–4 in Fig. 1, the multidimensional raw feature data table is produced and fuzzified as the fuzzy weighted data table, which is further aggregated into the single dimension data, a vector of Fuzzy Item Values (FIVs). To sort and rank items with FIVs, the best items are recommended in descending order. With taking FIVs to calculate the similarity by Grade Clustering algorithm, grade patterns are derived and can be used for recommendation. The details of each step of the FCPC-RGC are presented as follows.

### 2.1. Feature specification construction

When a new high-end product launches with new features introduced, customers may have no or little knowledge to the new features. The feature specification of the new product is designed and evaluated by enterprise domain experts. A product item comprises a set of features $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_N)$. Subject to the complexity of the item features structure, the item features can be organized as a hierarchical tree structure. Features in different levels are represented by nodes. A feature $\hat{\beta}_i$ has a set of $\hat{\beta}_i$'s sub-level features $\left(\hat{\beta}_{i,1}, \dots, \hat{\beta}_{i,j}, \dots, \hat{\beta}_{i,N_i}\right)$, and an sub-level feature $\hat{\beta}_{i,j}$ has a set of $\hat{\beta}_{i,j}$'s sub-level features $\left(\hat{\beta}_{i,j,1}, \dots, \hat{\beta}_{i,j,k}, \dots, \hat{\beta}_{i,j,N_{i,j}}\right)$. An example of the feature specification of smartphone is illustrated in Fig. 3 in Section 3.

### 2.2. Feature data fetch

The features organized as a hierarchical tree with several levels of nodes are regarded as a data scheme to fetch data. The internal nodes have the subordinate nodes whilst the external nodes are the leaf nodes without subordinate nodes. For each item, data for the measurable feature indicators$(c_1, \dots, c_j, \dots, c_N)$ located in the external nodes are fetched. Feature values in internal nodes are computed by using the measurable feature data. A raw data table of $M$ item sets of $N$ measureable features $\{r_{kj}: k = 1, \dots, M; j = 1, \dots, N;\}$ is created to structure the item data.

### 2.3. Feature data fuzzification

The raw data table $\{r_{kj}\}$ may contain crisp, nominal, ordinal and/or

Regarding the factors below to purchase a smartphone,
$\hat{\beta}_1$: Price Attractiveness,
$\hat{\beta}_2$: Brand,
$\hat{\beta}_3$: Product attributes ,
$\hat{\beta}_4$: Release Date
Compare the relative preference for each pair, and circle the mark accordingly.

| | Absolutely | | Significantly | | Highly | | Moderately | | Equally | | Moderately | | Highly | | Significantly | | Absolutely | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\hat{\beta}_1$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\hat{\beta}_2$ |
| $\hat{\beta}_1$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\hat{\beta}_3$ |
| $\hat{\beta}_1$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\hat{\beta}_4$ |
| $\hat{\beta}_2$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\hat{\beta}_3$ |
| $\hat{\beta}_2$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\hat{\beta}_4$ |
| $\hat{\beta}_3$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\hat{\beta}_4$ |

**Fig. 2.** A form for measuring the customer's relative consideration factors to buy a smartphone using fuzzy cognitive pairwise comparisons.

missing data and therefore cannot be directly processed by the proposed fuzzy algorithms. The raw data table $\{r_{kj}\}$ has to be preprocessed to a fuzzy data table $\{\widehat{r_{kj}}\}$ with $M$ fuzzy item sets of $N$ fuzzy measureable features. An fuzzy item set $\widehat{r_k}$ has a row vector of $N$ fuzzy feature values, i.e. $\widehat{r_k} = \{\widehat{r_{kj}}: j = 1, \ldots, N\}$, and there are $M$ rows of item sets, i.e. $k = 1, \ldots, M$.

Two methods to fuzzify the crisp data, additive fuzzification and multiplicative fuzzification, are proposed as below.

$$\widehat{r_{kj}} = F^{+}(r_{kj}, \delta^l, \delta^u) = (r_{kj} - \delta^l, r_{kj}, r_{kj} + \delta^u) \tag{1}$$

$$\widehat{r_{kj}} = F^{\times}(r_{kj}, \varepsilon^l, \varepsilon^u) = (r_j(1 - \varepsilon^l), r_{kj}, r_{kj}(1 + \varepsilon^u)),\ \ \varepsilon^l, \varepsilon^u \in [0, 1] \tag{2}$$

The additive fuzzification ($F^{+}$) function takes two boundary additive tuning parameters, $\delta^l$ and $\delta^u$, to convert a crisp number $r_{kj}$ into a triangular fuzzy number, i.e. $\widehat{r_{ij}} = (r_{ij}^l, r_{ij}^\pi, r_{ij}^u)$. The multiplicative fuzzification function, $F^{\times}$, takes two boundary ratio tuning parameters, $\varepsilon^l$and $\varepsilon^u$, to convert the crisp number $r_{ij}$ into a triangular fuzzy number $\widehat{r_{ij}}$. The choice for additive fuzzification and multiplicative fuzzification and the choice for tuning parameters for a selected fuzzification are subject to how an expert perceives a product feature value. An example of the choices is presented in Table 3 in Section 3.

If there are some missing data, the best practice is to manually fill the missing data by searching the related information from difference sources. Alternatively, some statistical methods may be used to estimate the values for missing data. In some cases, the items with missing data may be omitted. The discussion of missing data is beyond this research topic. If the data are nominal or ordinal, Fuzzy Cognitive Pairwise Comparison is proposed in the next step to map the data into the fuzzy numbers according to users' preferences.

### 2.4. Preference customization by FCPC

The feature data with nominal or ordinal linguistic labels cannot be directly computed unless numerical values for them are defined. Customized preferences can be represented by the numerical values. Fuzzy Cognitive Pairwise Comparison (FCPC) is the ideal method to calculate the fuzzy numbers to represent customized preferences. FCPC can also be used to determine feature weights of the decision criteria. The details of FCPC for the Preference Customization are presented as below.

To find the preferred products, a user fills an online form with searching criteria to express his/her preference. An example of an online form used in Section 3 is presented in Fig. 2. From the rating scores in the form submitted by the user, the Fuzzy Pairwise Opposite Matrices (FPOMs) are obtained. The Fuzzy Accordance Index (FAI) is applied to check the acceptance of each FPOM.

The fuzzy pairwise opposite matrix (FPOM) is used to interpret the individual utilities (weights or priorities) among a set of the evaluated objects. Fuzzy triangular number is chosen as fuzzy number due to its popularity in fuzzy applications. Let an ideal fuzzy utility set be $\widehat{V} = \{\widehat{v_1}, \ldots, \widehat{v_n}\}$, where the utility in fuzzy triangular number is of the form $\widehat{v_i} = (v_i^l, v_i^\pi, v_i^u)$, and the rating score in fuzzy number for comparison is $\widehat{b_{ij}} \cong \widehat{v_i} - \widehat{v_j}$. The ideal FPOM is $\widehat{\widetilde{\widetilde{B}}} = [\widehat{v_i} - \widehat{v_j}]$, and a judgmental FPOM using fuzzy paired interval scale is $\widehat{B} = [\widehat{b_{ij}}]$. $\widehat{\widetilde{\widetilde{B}}}$ is determined by $\widehat{B}$ as follows:

$$\widehat{\widetilde{\widetilde{B}}} = \left[\widehat{\widetilde{\widetilde{b_{ij}}}}\right] = \begin{bmatrix} (0,0,0) & \widehat{v_1} - \widehat{v_2} & \ldots & \widehat{v_1} - \widehat{v_n} \\ \widehat{v_2} - \widehat{v_1} & (0,0,0) & \ldots & \widehat{v_2} - \widehat{v_n} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{v_n} - \widehat{v_1} & \widehat{v_n} - \widehat{v_2} & \cdots & (0,0,0) \end{bmatrix} \cong \begin{bmatrix} \widehat{b_{11}} & \widehat{b_{12}} & \ldots & \widehat{b_{1n}} \\ \widehat{b_{21}} & \widehat{b_{22}} & \ldots & \widehat{b_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{b_{n1}} & \widehat{b_{n2}} & \cdots & \widehat{b_{nn}} \end{bmatrix}$$
$$= [\widehat{b_{ij}}] = \widehat{B} \tag{3}$$

$\widehat{b_{ij}} = (b_{ij}^l, b_{ij}^\pi, b_{ij}^u) = -\widehat{b_{ji}} = (-b_{ji}^u, -b_{ji}^\pi, -b_{ji}^l)$, and for $i, j = 1, \ldots, n$ and $i \neq j$. When $i = j$, then $\widehat{b_{ij}} = \widehat{v_i} - \widehat{v_j} = (0, 0, 0)$.

The indices $i$ and $j$ are local indices subject to their attached variable symbols. $\widehat{b_{ij}} \in \widehat{B}$ is a score rated and given by the users with the rating scale schema in Table 1. The crisp normal utility $\kappa$ is used to adjust a vector of scales in fuzzy triangular number. The rating scores can be organized as a fuzzy upper triangular matrix, and a lower triangular matrix is derived from the opposite of the upper triangular matrix. The number of comparisons in a FPOM is $\frac{n(n-1)}{2}$, $n \geq 2$. The validity of $\widehat{B}$ is tested by a Fuzzy Accordance Index $\widehat{AI}$ of the below form:

$\widehat{AI} = (AI^l)^{\frac{1}{4}} \times (AI^\pi)^{\frac{1}{2}} \times (AI^u)^{\frac{1}{4}}$, where

$$AI^l = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}^l,\ \ \delta_{ij}^l = \sqrt{Mean\left(\left(\frac{1}{\kappa^l}(B_i^l + (B_j^l)^T - b_{ij}^l)\right)^2\right)},\ \forall\, i, \forall\, j \in (1, \ldots, n),$$
$$AI^\pi = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}^\pi,\ \ \delta_{ij}^\pi = \sqrt{Mean\left(\left(\frac{1}{\kappa^\pi}(B_i^\pi + (B_j^\pi)^T - b_{ij}^\pi)\right)^2\right)},\ \forall\, i, \forall\, j \in (1, \ldots, n),$$
$$AI^u = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_{ij}^u,\ \ \delta_{ij}^u = \sqrt{Mean\left(\left(\frac{1}{\kappa^u}(B_i^u + (B_j^u)^T - b_{ij}^u)\right)^2\right)},\ \forall\, i, \forall\, j \in (1, \ldots, n); \tag{4}$$

$\widehat{\kappa} = (\kappa^l, \kappa^\pi, \kappa^u)$ is the fuzzy normal utility. By default, $\widehat{\kappa} = (\kappa^l, \kappa^\pi, \kappa^u) = (Max(\overline{X}_{\aleph}^\pi) - \delta, Max(\overline{X}_{\aleph}^\pi), Max(\overline{X}_{\aleph}^\pi) + \delta)$, and $\delta$ is the average of the modal values of two adjacent atomic terms, and $\overline{X}_{\aleph}^\pi$ is the set of the modal values from $\overline{X}_{\aleph}$.

$\widehat{AI}$, such that $\widehat{AI} \geq 0$, is the normalized weighted geometric mean of

**Table 1**
Rating scale schema for fuzzy cognitive pairwise comparisons.

| Scale | Notations | Fuzzy Paired Interval Scale |
|---|---|---|
| Equally | $\bar{0}$ | (0,0,0) |
| Slightly | $\bar{1}$ | $\left(0, \frac{\kappa}{8}, \frac{2\kappa}{8}\right)$ |
| Moderately | $\bar{2}$ | $\left(\frac{\kappa}{8}, \frac{2\kappa}{8}, \frac{3\kappa}{8}\right)$ |
| Fairly | $\bar{3}$ | $\left(\frac{2\kappa}{8}, \frac{3\kappa}{8}, \frac{4\kappa}{8}\right)$ |
| Highly | $\bar{4}$ | $\left(\frac{3\kappa}{8}, \frac{4\kappa}{8}, \frac{5\kappa}{8}\right)$ |
| Strongly | $\bar{5}$ | $\left(\frac{4\kappa}{8}, \frac{5\kappa}{8}, \frac{6\kappa}{8}\right)$ |
| Significantly | $\bar{6}$ | $\left(\frac{5\kappa}{8}, \frac{6\kappa}{8}, \frac{7\kappa}{8}\right)$ |
| Outstandingly | $\bar{7}$ | $\left(\frac{6\kappa}{8}, \frac{7\kappa}{8}, \frac{8\kappa}{8}\right)$ |
| Absolutely | $\bar{8}$ | $\left(\frac{7\kappa}{8}, \frac{8\kappa}{8}, \kappa\right)$ |

$(AI^l, AI^\pi, AI^u)$. If $\widehat{AI} = 0$, then $\widehat{B}$ is perfectly accordant; If $0 < \widehat{AI} \leq 0.1$, $\widehat{B}$ is accordant; If $\widehat{AI} > 0.1$, $\widehat{B}$ is unsatisfactory . The fuzzy priority/utility/weight/importance vector is derived from the FPOM by the Fuzzy Row Average plus the normal Utility *(FRAU)*(Yuen, 2009; Yuen, 2014a) of the form below.

$$FRAU(\widehat{B}, \widehat{\kappa}) = \left[ (v_i^l, v_i^\pi, v_i^u): \begin{pmatrix} v_i^l = \left(\frac{1}{n}\left(\sum_{j=1}^{i} b_{ij}^u + \sum_{j=i+1}^{n} b_{ij}^l\right)\right) + \kappa^l \\ v_i^\pi = \left(\frac{1}{n}\sum_{j=1}^{n} b_{ij}^\pi\right) + \kappa^\pi \\ v_i^u = \left(\frac{1}{n}\left(\sum_{j=1}^{i} b_{ij}^l + \sum_{j=i+1}^{n} b_{ij}^u\right)\right) + \kappa^u \end{pmatrix}, \forall\, i \in \{1, \ldots, n\} \right] \tag{5}$$

The normalized fuzzy priority vector is defined as $\widehat{W} = \{\widehat{w_1}, \ldots, \widehat{w_i}, \ldots, \widehat{w_n}\}$, where $\widehat{w_i} = (w_i^l, w_i^\pi, w_i^u)$ such that $\sum_{i=1}^{n} w_i^\pi = 1$. The fuzzy individual utility set from the FPOM is normalized as the fuzzy normalized priority vector by the normalization function as below.

$$\widehat{W} = NFRAU(\widehat{B}, \widehat{\kappa}) = Norm(FRAU(\widehat{B}, \widehat{\kappa})) = Norm(\widehat{V}) = \left\{ \begin{matrix} \widehat{w_i} = (w_i^l, w_i^\pi, w_i^u): \\ (w_i^l, w_i^\pi, w_i^u) = \left(\frac{v_i^l}{n\kappa^\pi}, \frac{v_i^\pi}{n\kappa^\pi}, \frac{v_i^u}{n\kappa^\pi}\right), \forall\, i \in \{1, \ldots, n\} \end{matrix} \right\},$$

where

$$\sum_{i \in \{1,\ldots,n\}} v_i^\pi = n\kappa^\pi \text{ and } \kappa^\pi = Max(\overline{X}_{\aleph}) \tag{6}$$

The feature specification is defined in a hierarchical tree structure presented in Section 2.1. If the fuzzy weights of the subordinate features of a feature can be assessed by a user, the proposed FCPC is the ideal tool to be used. For a FPOM for $\hat{\beta}_i$, i.e. $\widehat{B}_i$, the fuzzy weights of its sub-level features $\left\{\hat{\beta}_{i,j}\right\}$, i.e. $w\left(\left\{\hat{\beta}_{i,j}\right\}|\hat{\beta}_i\right)$, can be calculated as the form below.

$$w\left(\left\{\hat{\beta}_{i,j}\right\}|\hat{\beta}_i\right) = NFRAU(\widehat{B}_i, \widehat{\kappa}), \;\; i \in \{1, \ldots, N\} \tag{7}$$

Similarly, for a FPOM for $\hat{\beta}_{i,j}$, the fuzzy weights of its sub-level features $\left\{\hat{\beta}_{i,j,k}\right\}$ are the form below.

$$w\left(\left\{\hat{\beta}_{i,j,k}\right\}|\hat{\beta}_{i,j}\right) = NFRAU(\widehat{B}_{i,j}, \widehat{\kappa}), \;\; i \in \{1, \ldots, N\} \text{ and } j \in \{1, \ldots, N_i\} \tag{8}$$

In this paper, a tree for the feature specification is defined as up to four levels. Theoretically, more levels can be added and modeled by more subscript indices such as *i,j,k,l,n,m* and so on, but too many levels may not be practical in most real world applications. Up to four levels may be enough. If sub-level features under a feature are with numerical values, some other methods, e.g. normalization, may be applied to compute the fuzzy weights by directly taking the numerical values. Whilst the sub-level features under a feature are not with numerical values, the proposed FCPC is typically used to model and translate the user preferences into fuzzy numbers.

### 2.5. Fuzzy data fusion

The Fuzzy Data Fusion (FDF) operation is used to derive the Fuzzy Item Values (FIVs) by aggregating weighted feature values for each item. The FDP comprises two major parts: fuzzy normalization and fuzzy weighted aggregation. In fuzzy normalization process, fuzzy weights and fuzzy feature values are rescaled within the same fuzzy scale range, e.g. between (0,0,0) and (1,1,1). Without fuzzy normalization, the features of large values may significantly dominate the FIVs, and thus fuzzy normalization can offset the dominated bias. Two normalization methods are introduced with respect to whether the higher or lower value is preferred.

By fuzzy normalization functions, a raw fuzzy set $\widehat{r_{kj}} = (r_{kj}^l, r_{kj}^\pi, r_{kj}^u), \forall\, k \in \{1, \ldots, M\}, \forall\, j \in \{1, \ldots, N\}$ is rescaled to a normalized fuzzy set $\widehat{x_{kj}} = (x_{kj}^l, x_{kj}^\pi, x_{kj}^u)$. If the higher fuzzy value reflects the higher preference, the Dividing Maximal Function $\Delta_{\max}$ below is applied to rescale the raw fuzzy value.

$$\widehat{x_{kj}} = \Delta_{\max}(\widehat{r_{kj}}) = \frac{(r_{kj}^u, r_{kj}^\pi, r_{kj}^l)}{\max(\{(r_{kj}^l, r_{kj}^\pi, r_{kj}^u)\})}, \;\; \forall\, k \in \{1, \ldots, M\}, \;\; \forall\, j \in \{1, \ldots, N\} \tag{9}$$

If the lower fuzzy value is preferred, the Minimal Dividing Function $\Delta_{\min}$ below is used for the normalization.

$$\widehat{x_{kj}} = \Delta_{\min}(\widehat{r_{kj}}) = \frac{\min(\{(r_{kj}^l, r_{kj}^\pi, r_{kj}^u)\})}{(r_{kj}^u, r_{kj}^\pi, r_{kj}^l)}, \;\; \forall\, k \in \{1, \ldots, M\}, \;\; \forall\, j \in \{1, \ldots, N\} \tag{10}$$

The function $\max(\{(r_{kj}^l, r_{kj}^\pi, r_{kj}^u)\})$returns a highest crisp element value which is normally located in $\{r_{kj}^u\}$, whilst the function $\min(\{(r_{kj}^l, r_{kj}^\pi, r_{kj}^u)\})$ returns a lowest crisp element value which is normally located in $\{r_{kj}^l\}$.

A Normalized Fuzzy Weighted Data Table (NFWDT) illustrated in Table 2 comprises a matrix of normalized fuzzy feature values $\{\widehat{x_{kj}}\}$ with respect to an items vector $\widehat{T} = \{\widehat{T_1}, \ldots, \widehat{T_k}, \ldots, \widehat{T_M}\}$ and a measurable features vector $\widehat{C} = \{\widehat{c_1}, \ldots, \widehat{c_j}, \ldots, \widehat{c_N}\}$ associated with a vector of fuzzy measurable features weights $\widehat{\Lambda} = (\widehat{\lambda_1}, \ldots, \widehat{\lambda_j}, \ldots, \widehat{\lambda_N})$. A fuzzy feature value $\widehat{x_{kj}} = (x_{kj}^l, x_{kj}^\pi, x_{kj}^u)$ is a triangular fuzzy number for an item $\widehat{T_k}$ with respect to a measurable feature $\widehat{C_j}$. A fuzzy weight for $\widehat{C_j}$, i.e. $\widehat{\lambda_j} = (\lambda_j^l, \lambda_j^\pi, \lambda_j^u)$, is calculated as the form below.

$$\widehat{\lambda_j} = \prod_{h=H(\widehat{c_j})}^{2} \lambda(\widehat{c}_j^{\,h}|\widehat{c}_j^{\,h-1}), \;\; \forall\, j \in \{1, \ldots, N\} \tag{11}$$

The function $\lambda(\widehat{c}_j^{\,h}|\widehat{c}_j^{\,h-1})$ returns the fuzzy weight of a feature node $\widehat{c}_j^{\,h}$ at level $h$ under its parent $\widehat{c}_j^{\,h-1}$ of the tree which the fuzzy weights under each parent node are derived by methods presented in Section 2.4. $h$ is index of hierarchical levels. When $h$=1, there is a single node at the top level, i.e. level 1. The function $H(\widehat{c_j})$ returns the number of hierarchical levels of a measurable feature $\widehat{c_j}$. Eq. (9) is the chain

**Table 2**
A normalized fuzzy weighted data table with fuzzy Item values.

| Item | $\widehat{C}_1$ $(\lambda_1^l, \lambda_1^\pi, \lambda_1^u)$ | ⋯ | $\widehat{C}_j$ $(\lambda_j^l, \lambda_j^\pi, \lambda_j^u)$ | ⋯ | $\widehat{C}_N$ $(\lambda_N^l, \lambda_N^\pi, \lambda_N^u)$ | $FIV\ (\widehat{X}_k)$ |
|---|---|---|---|---|---|---|
| $\widehat{T}_1$ | $(x_{11}^l, x_{11}^\pi, x_{11}^u)$ | ⋯ | $(x_{1j}^l, x_{1j}^\pi, x_{1j}^u)$ | ⋯ | $(x_{1N}^l, x_{1N}^\pi, x_{1N}^u)$ | $(x_1^l, x_1^\pi, x_1^u)$ |
| ⋮ | ⋮ | ⋯ | | ⋯ | ⋮ | ⋮ |
| $\widehat{T}_k$ | $(x_{k1}^l, x_{k1}^\pi, x_{k1}^u)$ | ⋯ | $(x_{kj}^l, x_{kj}^\pi, x_{kj}^u)$ | ⋯ | $(x_{kN}^l, x_{kN}^\pi, x_{kN}^u)$ | $(x_k^l, x_k^\pi, x_k^u)$ |
| ⋮ | ⋮ | ⋯ | | ⋯ | ⋮ | ⋮ |
| $\widehat{T}_M$ | $(x_{M1}^l, x_{M1}^\pi, x_{M1}^u)$ | ⋯ | $(x_{Mj}^l, x_{Mj}^\pi, x_{Mj}^u)$ | ⋯ | $(v_{MN}^l, v_{MN}^\pi, v_{MN}^u)$ | $(x_M^l, x_M^\pi, x_M^u)$ |

products of the fuzzy weights from leaves to the top node. The element wise multiplication is applied to the multiplication of two fuzzy triangular numbers.
$\{\widehat{x_{kj}}\}$ and $\widehat{\Lambda}$ are aggregated to a set of Fuzzy Item Values $\{\widehat{X}_k\}$ by a fuzzy aggregation operator. Fuzzy Arithmetic Mean (FAM) of the form below is chosen as default fuzzy aggregation operator due to its popularity, computational efficiency and comprehensive simplicity.

$$\widehat{X}_k = \left\{(x_k^l, x_k^\pi, x_k^u): \left(\begin{array}{l} x_k^l = \frac{1}{N}\sum_{j=1}^N x_{kj}^l\lambda_j^l \\ x_k^\pi = \frac{1}{N}\sum_{j=1}^N x_{kj}^\pi\lambda_j^\pi \\ x_k^u = \frac{1}{N}\sum_{j=1}^N x_{kj}^u\lambda_j^u \end{array}\right)\right\}, k = 1, \ldots, M. \tag{12}$$

## 2.6. Items sorting and ranking

As the Fuzzy Item Value (FIV) is of a fuzzy triangular number (FTN) which cannot directly be determined to rank the items, the centroid defuzzification operator is used to defuzzify a fuzzy number into a conventional crisp number. A Defuzzified Item Value ($DIV$),$\alpha_k$, is derived from the centroid defuzzification of a FIV in FTN of the form below.

$$\alpha_k = DIV(\widehat{X}_k) = \frac{1}{3}(x_k^l + x_k^\pi + x_k^u), k = 1, \ldots, M \tag{13}$$

The above function defines the values $\{\alpha_1, \ldots, \alpha_k, \ldots \alpha_M\}$ as the references to be sorted. $\{\theta_1, \ldots, \theta_s, \ldots, \theta_M\}$ is a vector of sorted items from $\{\alpha_k\}$. $k$ is the index of an unsorted item and $s$ is the index of a sorted item. In order to rank the items, the general form of a sorting algorithm is defined as follow.

$$\theta = \{(\theta_s, k)\} = sort(\{\alpha_k, k\}) = sort(\{\alpha_k\}, \{k\}),$$
$$s = 1, \ldots, M \text{ and } \forall\ k \in K = \{1, \ldots, M\} \tag{14}$$

The above function returns a set of pair values $\{(\theta_s, k)\}$. $\theta_s$ is usually ordered in the property defined below.

$$\theta_s = \alpha_k \text{ such that } \theta_{s-1} \geq \theta_s, s = 2, \ldots, M \text{ and } \forall\ k \in \{1, \ldots, M\} \tag{15}$$

**Algorithm 1.** Selection Sort For Items Ranking (SSFIR).

*SelectionSortForItemsRanking*$(\{\alpha_k\}, \{k\})$

```
{θ_s} = {α_k}   //initiate a vector of item values
K = {k}   //initiate a vector of item indices
For i = 1 to M–1 do
    I+ = i;   // define index for max number in each iteration
    For j = i+1 to M do
        If θ_{I+} < θ_j, then I+ = j;
    If i ≠ I+, then
        Exchange θ_{I+} and θ_i;
        Exchange K_{I+} and K_i;
Return ({θ_s}, K)
```

The above form means that the higher $DIV$ has the higher rank. A lot of well-established sorting algorithms can be used to implement the generate forms of Eqs. (14) and (15), for example, selection sort, insertion sort, merge sort, bubble sort, heapsort, quicksort, and so on. As the discussion of sorting algorithm is beyond the topic of this research, the interested readers can learn about the popular sorting methods in (Cormen, 2009; Levitin and Mukherjee, 2003) . For the simplicity, this research applies selection sort to implement the generate forms of Eqs. (14) and (15). Whilst the classical selection sort taking a single vector of data does not take a pair of vectors, i.e. item values $\{\alpha_k\}$ and the corresponding item numbers $\{k\}$, some modifications are needed for the query from the proposed recommender system. The item numbers are unique and essential for the search query, although they shows in order for the demonstration of this paper. The proposed Selection Sort For Items Ranking is presented in Algorithm 1.

## 2.7. Fuzzy grade clustering

The Fuzzy Grade Clustering,$FGC(\{\widehat{X}_k\}, G)$, is used to grade the items scored by $\{\widehat{X}_k\}$ into $G$ grading clusters, i.e. $U_g$, $g \in \{1, \ldots, G\}$. The grading cluster number $G$ is the number of ordered clusters such that the cluster with higher index number is of higher grade value. If $\{\widehat{X}_k\}$ is replaced by $\{\theta_s\}$, the cluster results are with sorted patterns for each cluster with respect to $DIV$. Extra step is required for mapping sorted index $s$ to unsorted index $k$. The computational steps of FGC are as follow.

Step i: Initializing grading centers

A set of initial ordered grading centers, $\widehat{\mu}^{(0)} = \left\{\widehat{\mu}_g^{(0)} = \left(\mu_g^l, \mu_g^\pi, \mu_g^u\right)^{(0)} : g \in \{1, \ldots, G\}\right\}$, is initialized with respect to $\{\widehat{X}_k\} = \{(x_k^l, x_k^\pi, x_k^u)\}$, the fuzzy average intervals between two adjacent grade centers are computed as below.

$$\rho^l = \frac{\max(\{x_k^l\}) - \min(\{x_k^l\})}{2G}$$
$$\rho^\pi = \frac{\max(\{x_k^\pi\}) - \min(\{x_k^\pi\})}{2G}$$
$$\rho^u = \frac{\max(\{x_k^u\}) - \min(\{x_k^u\})}{2G} \tag{16}$$

A set of fuzzy initial centers,$u^{(0)} = \left\{\widehat{\mu}_1^{(0)}, \ldots, \widehat{\mu}_g^{(0)}, \ldots, \widehat{\mu}_G^{(0)}\right\}$, is used to calculate the even distribution of the data into $G$ groups. The superscript, ($r$), indicates the number of the update states of the grade centers. When $r$=0, which means the initial state, a set of the fuzzy initial centers is calculated as below.

$$\widehat{\mu}_g^{(0)} = \left[\left(\mu_g^{l(0)}, \mu_g^{\pi(0)}, \mu_g^{u(0)}\right): \left(\begin{array}{l} \mu_g^{l(0)} = \min(\{x_k^l\}) + \rho^l(1 + 2(g-1)) \\ \mu_g^{\pi(0)} = \min(\{x_k^\pi\}) + \rho^\pi(1 + 2(g-1)) \\ \mu_g^{u(0)} = \min(\{x_k^u\}) + \rho^u(1 + 2(g-1)) \end{array}\right)\right]$$
$$\text{, where } g = 1, \ldots, G \tag{17}$$

Step ii: Computing distance matrix

The fuzzy distance matrix $\widehat{D}^{(r)} = \left\{\widehat{d}_{kg}^{(r)}\right\}$ is used to calculate a matrix of the distances between each $\widehat{X}_k$ and each fuzzy grading center $\widehat{\mu}_g^{(r)}$. The measurement of fuzzy distance,$\widehat{d}_{kg}^{(r)} = (d_{kg}^{l(r)}, d_{kg}^{\pi(r)}, d_{kg}^{u(r)})$, is calculated as below.

$$\widehat{d}_{kg}^{(r)} = (d_{kg}^{l(r)}, d_{kg}^{\pi(r)}, d_{kg}^{u(r)}) = \left(\sqrt{\left(x_k^l - \mu_g^{l(r)}\right)^2}, \sqrt{\left(x_k^\pi - \mu_g^{\pi(r)}\right)^2}, \sqrt{\left(x_k^u - \mu_g^{u(r)}\right)^2}\right),\quad g = 1, \dots, G \text{ and } k = 1, \dots, M \tag{18}$$

Step iii: Indexing and assigning items to clusters

On the basis of each row in the fuzzy distance matrix $\widehat{D}^{(r)}$, each $\widehat{X}_k$ is assigned to the fuzzy grading cluster $\widehat{U}_g^{(r)}$ of the grading center $\widehat{\mu}_g^{(r)}$ by the function below.

$$\widehat{U}_{g^*}^{(r)} = \widehat{U}_{g^*}^{(r-1)} \cup \{\widehat{X}_k\}$$
$$\text{such that } g^* = \arg\min\left(\left\{\widehat{d}_{kg}^{(r)}: g = 1, \dots, G\right\}\right), \forall\, k \in \{1, \dots, M\} \tag{19}$$

$\arg\min\left(\left\{\widehat{d}_{kg}^{(r)}: g = 1, \dots, G\right\}\right)$ returns the index $g^*$ of the grading cluster which $\widehat{X}_k$ belongs to, on the basis of the least distance $\widehat{d}_{kg}^{(r)}$ from $\widehat{X}_k$ to $\mu_g^{(r)}$.

To measure the least fuzzy distance, the centroid defuzzification for $\widehat{\alpha}_k$ in fuzzy triangular number is computed by the form below.

$$d_{kg}^{\prime(r)} = \frac{1}{3}(d_{kg}^{l(r)} + d_{kg}^{\pi(r)} + d_{kg}^{u(r)}) \tag{20}$$

Eq. (19) can be substituted by Eq. (20) to have the form below.

$$\widehat{U}_{g^*}^{(r)} = \widehat{U}_{g^*}^{(r)} \cup \{\widehat{X}_k\}$$
$$\text{such that } g^* = \arg\min(\{d_{kg}^{\prime(r)}: g = 1, \dots, G\}), \forall\, k \in \{1, \dots, M\} \tag{21}$$

Step iv: Updating grading centers

To update the grading centers, the iteration index $r$ increases one in each iteration, i.e. $r=r+1$. In other words, the index for the current elements is $r-1$. The new grading centers are recalculated on the basis of the mean value of the items in a cluster.

$$\widehat{\mu}_g^{(r)} = mean\left(\widehat{U}_g^{(r-1)}\right),\ \forall\, g \in \{1, \dots, G\}, \tag{22}$$

Explicitly,

$$\widehat{\mu}_g^{(r)} = \frac{1}{\left|\widehat{U}_g^{(r-1)}\right|}\sum_{s'=1}^{\left|\widehat{U}_g^{(r-1)}\right|}\widehat{X}_{s'},\ \text{where } \widehat{X}_{s'} \in \widehat{U}_g^{(r)},\ g = 1, \dots, G \tag{23}$$

$\left|\widehat{U}_g^{(r)}\right|$ returns the number of items in $\widehat{U}_g^{(r)}$.

Step v: Iteration and termination

Steps ii to v are repeated with $R$ iterations until the fuzzy grading centers at the current state has no further change, e.g. $\widehat{\mu}_g^{(r)} \equiv \widehat{\mu}_g^{(r-1)}$. The termination function is defined as below.

$$Max\left(\left\{\left(\widehat{\mu}_g^{(r)} - \widehat{\mu}_g^{(r-1)}\right)^2\right\}\right) = 0 \tag{24}$$

If Eq. (24) is satisfied, $\widehat{d}_{sg}^{(r)} \equiv \widehat{d}_{sg}^{(r-1)}$, and $\widehat{U}_g^{(r)} \equiv \widehat{U}_g^{(r-1)}$.

Step vi: Output

The grading cluster result of the last iteration will return.

$$\{\widehat{U}_g\} = \left\{\widehat{U}_g^{(r)}\right\} \tag{25}$$

### 2.8. Item recommendation

Supposed a user submits his online search form including his preference, and then chooses an item $\phi$ such that $\widehat{X}_\phi \in \widehat{U}_g$, for evaluation. The system will recommend the rest members in the same cluster, i.e.$Z$, for the user.

$$Z = \widehat{U}_g/\{\widehat{X}_\phi\},\ \text{where / is a complement operator.} \tag{26}$$

## 3. Application to smartphone recommendation system

To demonstrate the validity and applicability of FCPC-RGC, the proposed approach is applied to build a smartphone recommendation system. When a consumer is said to buy a product, the consumer buys a bundle of features rather than the product itself. The consumer expects a phone not only for talking, but also for many other functions such as camera, games, books, movies, and music. Due to many smartphones selling in the market, it is really time-consuming for the consumers to read and search rich information and compare vast alternatives one by one. A smartphone retail shop plans to establish an online recommender system at its current online store to capture the consumers' preferences and reduce the consumers' time to find the most suitable smartphones among the latest alternatives. The smartphone recommender system applying the proposed FCPC-RGC is presented as follows.

### 3.1. Feature specification construction

The stakeholders from purchasing, marketing and engineering departments define the evaluation schema for the distinct smartphone features. To reduce the evaluation workloads, only distinct product features are selected for evaluations and comparisons. Some basic features, such as the latest versions of Wifi, GPS, and Bluetooth, are not considered as almost all recent smartphone devices have such expected functions. After the comprehensive feature definition process, the smartphone feature specification, the data schema for fetching data, is shown in Fig. 3.

A smartphone for the purchasing decision is represented by four categories of major features: price attractiveness ($\hat{\beta}_1$), brand ($\hat{\beta}_2$), smartphone device ($\hat{\beta}_3$), and release date ($\hat{\beta}_4$). The smartphone device ($\hat{\beta}_3$) is measured by operating system ($\hat{\beta}_{3,1}$), processer speed ($\hat{\beta}_{3,2}$), display quality ($\hat{\beta}_{3,3}$), size and weight ($\hat{\beta}_{3,4}$), memory ($\hat{\beta}_{3,5}$), camera ($\hat{\beta}_{3,6}$), battery life ($\hat{\beta}_{3,7}$). Some features,$\hat{\beta}_{3,4}$,$\hat{\beta}_{3,5}$,$\hat{\beta}_{3,6}$ and $\hat{\beta}_{3,7}$, are further measured by their subordinate features.

### 3.2. Feature data fetch

Twenty popular smartphones in Hong Kong market in Feb 2015 were selected for the purpose of the reproducible research demonstrating calculation steps of the proposed methods. The features data could be obtained from the manufacturers, online retails shops or magazines. The historical data of the old models should not be considered for the mining process as the features are out-of-date and will not be sold in the market anymore. The setting details of the measurable features, which are the external nodes of the tree shown in Fig. 3, are presented in Table 3. The raw data table is presented in Table A1 in .

### 3.3. Feature data fuzzification

A crisp value may be less appropriate to represent a feature value. For example, information for some features may be different from different sources such as manufacturers, retailers, magazines, third party testing agents, or company's testers. Some features such as price

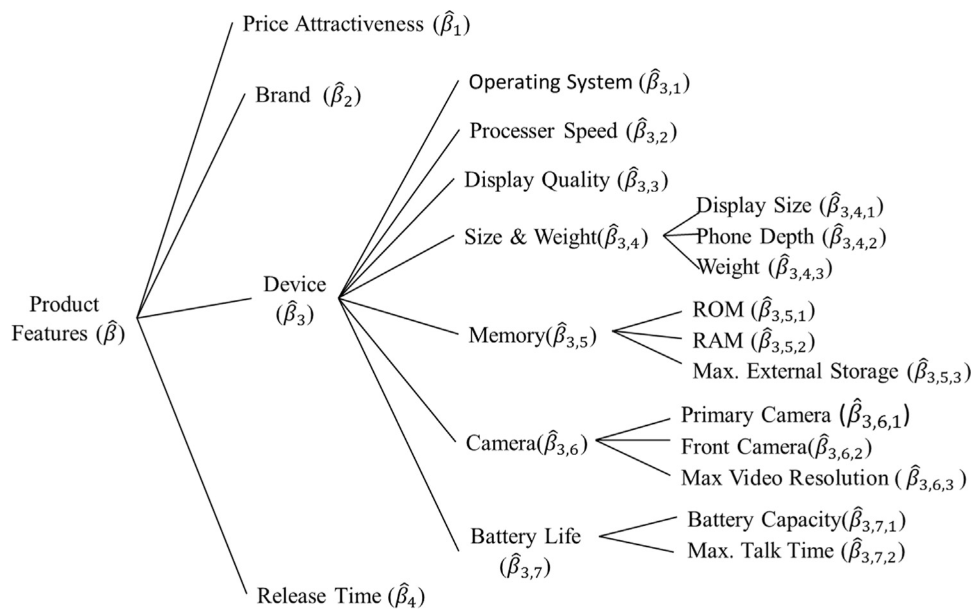


**Fig. 3.** Structure of Smartphone Feature Specification.

and brand may have a range of the values. The data in fuzzy number should be more appropriate to represent the uncertainty issue of the range of the values. The fuzzification methods using Eqs. (1) and (2) for each measurable feature are shown in Table 3. By fuzzifying the raw table shown in Table A1 in Appendix, the smartphone feature data in fuzzy number are presented in Table A2 and A3 in Appendix. For example, for the display size ($c_6$) and price ($c_1$) of Item 1, the fuzzy numbers by using Eqs. (1) and (2) respectively are calculated as below.

$$\widehat{r_{1,6}} = F^{+}(r_{1,6}, 0.2, 0.2) = (4, 0.2, 0.2) = (3.8, 4, 4.2)$$

$$\widehat{r_{1,1}} = F^{\times}(4688, 0.9, 1.05) = (4688{\cdot}0.9, 4688, 4688{\cdot}1.05) = (4219, 4688, 4922)$$

The advantage of fuzzying the data is that the crisp number is not fixed. For example, product price may be different in different shops and screen sizes may have some variance from the described standard size. Such uncertainty for a feature value could be represented in fuzzy number. The choices of functions and parameters are defined by the domain experts.

### 3.4. Preference customization by FCPC

As customers have different preferences for the features, elicitation of the customer preferences is a challenging task. The perception of the preference importance among alternatives is a relative concept and can be measured by fuzzy cognitive pairwise comparisons, which should work better than the conventional direct rating method. Table 4 illustrates the rating scale schema for FCPC. Forward comparison means that object A is better than object B, whilst backward comparison means that object A is worse than object B. The scale values of the backward comparison are opposite to the scale values of the forward

**Table 3**
Schema of measurable features of a smartphone.

| Notation | Measurable feature | Measurement scale | Fuzzification | Normalization |
|---|---|---|---|---|
| $c_1$ | Price Attractiveness ($\hat{\beta}_1$) | HKD | $F^{\times}(r_{i,1}, 0.9, 1.05)$ | $\Delta_{\min}$ |
| $c_2$ | Brand ($\hat{\beta}_2$) | Nominal: Apple, Samsung, LG, HTC, and Sony | FCPC | $\Delta_{\max}$ |
| $c_3$ | Operating System ($\hat{\beta}_{3,1}$) | Nominal: iOS and Android | FCPC | $\Delta_{\max}$ |
| $c_4$ | Processer Speed ($\hat{\beta}_{3,2}$) | Geekbench Multi-Core testing score | $F^{\times}(r_{i,4}, 0.9, 1.1)$ | $\Delta_{\max}$ |
| $c_5$ | Display Quality ($\hat{\beta}_{3,3}$) | Pixel Density (PPI) | $F^{\times}(r_{i,5}, 0.9, 1.05)$ | $\Delta_{\max}$ |
| $c_6$ | Display Size ($\hat{\beta}_{3,4,1}$) | Inch | $F^{+}(r_{i,6}, 0.2, 0.2)$ | $\Delta_{\max}$ |
| $c_7$ | Phone Depth ($\hat{\beta}_{3,4,2}$) | mm | $F^{+}(r_{i,7}, 0.2, 0.2)$ | $\Delta_{\max}$ |
| $c_8$ | Weight ($\hat{\beta}_{3,4,3}$) | Gram | $F^{+}(r_{i,8}, 2, 5)$ | $\Delta_{\min}$ |
| $c_9$ | ROM ($\hat{\beta}_{3,5,1}$) | Gigabyte | $F^{\times}(r_{i,9}, 0.9, 1.05)$ | $\Delta_{\max}$ |
| $c_{10}$ | RAM ($\hat{\beta}_{3,5,2}$) | Gigabyte | $F^{\times}(r_{i,10}, 0.9, 1.05)$ | $\Delta_{\max}$ |
| $c_{11}$ | Max. external storage ($\hat{\beta}_{3,5,3}$) | Gigabyte | $F^{\times}(r_{i,11}, 0.5, 1)$ | $\Delta_{\max}$ |
| $c_{12}$ | Primary Camera ($\hat{\beta}_{3,6,1}$) | megapixels | $F^{\times}(r_{i,12}, 0.9, 1)$ | $\Delta_{\max}$ |
| $c_{13}$ | Front Camera ($\hat{\beta}_{3,6,2}$) | megapixels | $F^{\times}(r_{i,13}, 0.9, 1)$ | $\Delta_{\max}$ |
| $c_{14}$ | Max Video Resolution ($\hat{\beta}_{3,6,3}$) | Ordinal scales of megapixels: 1080p, and 4 k | 1080p:{0.2,0.25,0.3}<br>4k: {0.9,0.95,1} | |
| $c_{15}$ | Battery Capacity ($\hat{\beta}_{3,7,1}$) | Hours | $F^{\times}(r_{i,15}, 0.9, 1.05)$ | $\Delta_{\max}$ |
| $c_{16}$ | Max. Talk Time ($\hat{\beta}_{3,7,2}$) | Hours | $F^{\times}(r_{i,14}, 0.9, 1)$ | $\Delta_{\max}$ |
| $c_{17}$ | Release Date ($\hat{\beta}_4$) | The earliest one is defined as 1. One month period is one unit. | $F^{+}(r_{i,17}, 1, 1)$ | $\Delta_{\max}$ |

**Table 4**
Rating scale schema for FCPC.

| Level of verbal scale | Forward comparison: Notation | Forward comparison: Fuzzy Paired Interval Scale | Backward comparison: Notation | Backward comparison: Fuzzy Paired Interval Scale |
|---|---|---|---|---|
| Equally | $\bar{0}$ | (0,0,0) | $\bar{0}$ | (0,0,0) |
| Slightly | $\bar{1}^+$ | (0,1,2) | $\bar{1}^-$ | (−2,−1,0) |
| Moderately | $\bar{2}^+$ | (1,2,3) | $\bar{2}^-$ | (−3,−2,−1) |
| Fairly | $\bar{3}^+$ | (2,3,4) | $\bar{3}^-$ | (−4,−3,−2) |
| Highly | $\bar{4}^+$ | (3,4,5) | $\bar{4}^-$ | (−5,−4,−3) |
| Strongly | $\bar{5}^+$ | (4,5,6) | $\bar{5}^-$ | (−6,−5,−4) |
| Significantly | $\bar{6}^+$ | (5,6,7) | $\bar{6}^-$ | (−7,−6,−5) |
| Outstandingly | $\bar{7}^+$ | (6,7,8) | $\bar{7}^-$ | (−8,−7,−6) |
| Absolutely | $\bar{8}^+$ | (7,8,8) | $\bar{8}^-$ | (−8,−8,−7) |

**Table 5**
Fuzzy comparison matrix for customer's relative preferences to buy a smartphone.

| $\widehat{B}_0$ | Price attractiveness ($\hat{\beta}_1$) | Brand ($\hat{\beta}_2$) | Smartphone device ($\hat{\beta}_3$) | Release date ($\hat{\beta}_4$) |
|---|---|---|---|---|
| $\widehat{\beta}_1$ | $\bar{0}$ | $\bar{8}^+$ | $\bar{2}^+$ | $\bar{5}^+$ |
| $\widehat{\beta}_2$ | $\bar{8}^-$ | $\bar{0}$ | $\bar{7}^-$ | $\bar{4}^-$ |
| $\widehat{\beta}_3$ | $\bar{2}^-$ | $\bar{7}^+$ | $\bar{0}$ | $\bar{5}^+$ |
| $\widehat{\beta}_4$ | $\bar{5}^-$ | $\bar{4}^+$ | $\bar{5}^-$ | $\bar{0}$ |
| $\widehat{AI}$ =0.1 | | | | |

comparison. Fig. 2 shows a form using FCPC to measure customer's relative preferences to buy a smartphone. The rating scores are presented in a matrix shown in Table 5. Table 6 presents the fuzzy comparison matrices for the sub-level features of the product features.

Some feature data of nominal scale are unable to be directly clustered unless they are specified as fuzzy numerical values. For example, no specific numerical values represent brand values such as Apple, Samsung, LG, HTC, and Sony. Similarly, label values for OS such as iOS and Android cannot be directly used for calculation. To convert the nominal labels to numerical values, FCPC is applied to elicit the preferences of such nominal features for fuzzy values. Table 7 presents the fuzzy comparison matrices from an individual user to measure the preference values for brand and OS features respectively. All fuzzy comparison matrices are validated by the measurement of

**Table 7**
Fuzzy comparison matrices ($\widehat{B}_8$ to $\widehat{B}_9$) for nominal features of Brand and OS.

| $\widehat{B}_8$ | Apple | Samsung | LG | HTC | Sony |
|---|---|---|---|---|---|
| Apple | $\bar{0}$ | $\bar{3}^+$ | $\bar{4}^+$ | $\bar{5}^+$ | $\bar{4}^+$ |
| Samsung | $\bar{3}^-$ | $\bar{0}$ | $\bar{1}^+$ | $\bar{2}^+$ | $\bar{1}^+$ |
| LG | $\bar{4}^-$ | $\bar{1}^-$ | $\bar{0}$ | $\bar{1}^+$ | $\bar{1}^+$ |
| HTC | $\bar{5}^-$ | $\bar{2}^-$ | $\bar{1}^-$ | $\bar{0}$ | $\bar{1}^-$ |
| Sony | $\bar{4}^-$ | $\bar{1}^-$ | $\bar{1}^-$ | $\bar{1}^+$ | $\bar{0}$ |
| $\widehat{AI}$ =0.049 | | | | | |

| $\widehat{B}_9$ | iOS | Android |
|---|---|---|
| iOS | $\bar{0}$ | $\bar{6}^+$ |
| Android | $\bar{6}^-$ | $\bar{0}$ |
| $\widehat{AI}$ =0 | | |

Fuzzy Accordance Index $\widehat{AI}$ (Eq. (4)).

By using Eqs. (5)–(8), the fuzzy comparison matrices (Tables 5 and 6) are prioritized and normalized as the fuzzy weights for smartphone features presented in Fig. 4. The fuzzy weight of a higher level feature is the fuzzy arithmetic mean of the fuzzy weights of its subordinate features. Only the fuzzy weights of measurable features will be used in the next step. The aggregated weights for the internal nodes can be used for the references to compare the preferences at the same levels.

### 3.5. Fuzzy data fusion

Fuzzy data fusion comprises fuzzy normalization and fuzzy aggregation. Fuzzy normalization is used to offset the dominated bias induced by the features of large values for the aggregation to the Fuzzy Item Values (FIVs). The normalization methods settings using $\Delta_{\max}$ or $\Delta_{\min}$ (Eqs. (9) and (10)) for each feature are given respectively in Table 3. $\Delta_{\max}$ is used for that the larger feature values are more preferred whilst $\Delta_{\min}$ is used for the otherwise. The normalized fuzzy feature data are shown in Tables A4 & A5 in Appendix. For example, for the display size and price of product item 1, the normalized fuzzy feature data by using Eqs. (9) and (10) respectively are calculated as below.

$$\widehat{x}_{16} = \Delta_{\max}((3.8,\ 4,\ 4.2)) = \frac{(3.8,\ 4,\ 4.2)}{\max(\{\widehat{r_{kj}}\})} = \frac{(3.8,\ 4,\ 4.2)}{5.9}$$

$$= (0.644,\ 0.678,\ 0.712)$$

$$\widehat{x}_{11} = \Delta_{\min}((4919,\ 4688,\ 4222)) = \frac{2338}{(4922,\ 4688,\ 4219)}$$

$$= (0.475,\ 0.499,\ 0.554)$$

The aggregation results of the fuzzy weights of the measurable features $\{\widehat{\lambda}_j\}$ are derived by Eq. (11) and presented in Fig. 4. To further illustrate the calculation of Eq. (11), the fuzzy weight $\widehat{\lambda}_6$ is taken as

**Table 6**
Fuzzy comparison matrices ($\widehat{B}_3$ to $\widehat{B}_7$) for sub-level features under feature $\widehat{\beta}_3$.

| $\widehat{B}_3$ | $\widehat{\beta}_{3,1}$ | $\widehat{\beta}_{3,2}$ | $\widehat{\beta}_{3,3}$ | $\widehat{\beta}_{3,4}$ | $\widehat{\beta}_{3,5}$ | $\widehat{\beta}_{3,6}$ | $\widehat{\beta}_{3,7}$ |
|---|---|---|---|---|---|---|---|
| $\widehat{\beta}_{3,1}$ | $\bar{0}$ | $\bar{6}^-$ | $\bar{5}^-$ | $\bar{4}^-$ | $\bar{7}^-$ | $\bar{7}^-$ | $\bar{7}^-$ |
| $\widehat{\beta}_{3,2}$ | $\bar{6}^+$ | $\bar{0}$ | $\bar{2}^+$ | $\bar{3}^+$ | $\bar{1}^-$ | $\bar{1}^-$ | $\bar{1}^-$ |
| $\widehat{\beta}_{3,3}$ | $\bar{5}^+$ | $\bar{2}^-$ | $\bar{0}$ | $\bar{2}^-$ | $\bar{3}^-$ | $\bar{3}^-$ | $\bar{3}^-$ |
| $\widehat{\beta}_{3,4}$ | $\bar{4}^+$ | $\bar{3}^-$ | $\bar{2}^+$ | $\bar{0}$ | $\bar{2}^-$ | $\bar{3}^-$ | $\bar{3}^-$ |
| $\widehat{\beta}_{3,5}$ | $\bar{7}^+$ | $\bar{1}^+$ | $\bar{3}^+$ | $\bar{2}^+$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ |
| $\widehat{\beta}_{3,6}$ | $\bar{7}^+$ | $\bar{1}^+$ | $\bar{3}^+$ | $\bar{3}^+$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ |
| $\widehat{\beta}_{3,7}$ | $\bar{7}^+$ | $\bar{1}^+$ | $\bar{3}^+$ | $\bar{3}^+$ | $\bar{0}$ | $\bar{0}$ | $\bar{0}$ |
| $\widehat{AI}$ =0.084 | | | | | | | |

| $\widehat{B}_4$ | $\widehat{\beta}_{3,4,1}$ | $\widehat{\beta}_{3,4,2}$ | $\widehat{\beta}_{3,4,3}$ |
|---|---|---|---|
| $\widehat{\beta}_{3,4,1}$ | $\bar{0}$ | $\bar{4}^+$ | $\bar{3}^+$ |
| $\widehat{\beta}_{3,4,2}$ | $\bar{4}^-$ | $\bar{0}$ | $\bar{2}^-$ |
| $\widehat{\beta}_{3,4,3}$ | $\bar{3}^-$ | $\bar{2}^+$ | $\bar{0}$ |
| $\widehat{AI}$ = 0 | | | |

| $\widehat{B}_5$ | $\widehat{\beta}_{3,5,1}$ | $\widehat{\beta}_{3,5,2}$ | $\widehat{\beta}_{3,5,3}$ |
|---|---|---|---|
| $\widehat{\beta}_{3,5,1}$ | $\bar{0}$ | $\bar{1}^+$ | $\bar{3}^-$ |
| $\widehat{\beta}_{3,5,2}$ | $\bar{1}^+$ | $\bar{0}$ | $\bar{2}^-$ |
| $\widehat{\beta}_{3,5,3}$ | $\bar{3}^+$ | $\bar{2}^+$ | $\bar{0}$ |
| $\widehat{AI}$ =0.08 | | | |

| $\widehat{B}_6$ | $\widehat{\beta}_{3,6,1}$ | $\widehat{\beta}_{3,6,2}$ | $\widehat{\beta}_{3,6,3}$ |
|---|---|---|---|
| $\widehat{\beta}_{3,6,1}$ | $\bar{0}$ | $\bar{3}^+$ | $\bar{5}^+$ |
| $\widehat{\beta}_{3,6,2}$ | $\bar{3}^-$ | $\bar{0}$ | $\bar{2}^+$ |
| $\widehat{\beta}_{3,6,3}$ | $\bar{5}^-$ | $\bar{2}^-$ | $\bar{0}$ |
| $\widehat{AI}$ =0.78 | | | |

| $\widehat{B}_7$ | $\widehat{\beta}_{3,7,1}$ | $\widehat{\beta}_{3,7,2}$ |
|---|---|---|
| $\widehat{\beta}_{3,7,1}$ | $\bar{0}$ | $\bar{4}^-$ |
| $\widehat{\beta}_{3,7,2}$ | $\bar{4}^+$ | $\bar{0}$ |
| $\widehat{AI}$ =0 | | |

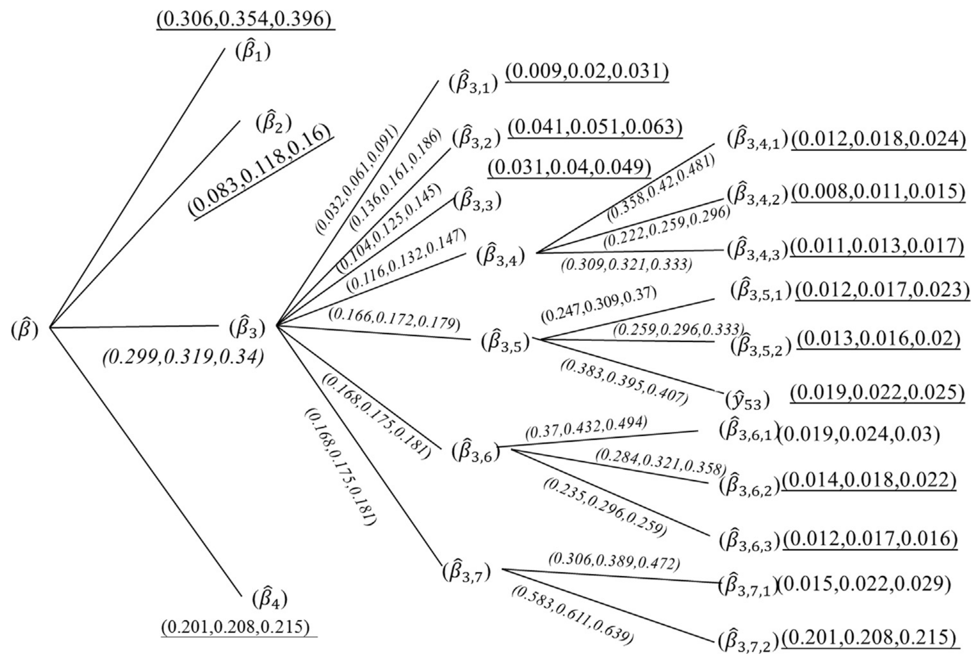


**Fig. 4.** Smartphone feature weights in fuzzy number.

**Table 8**
Results of items sorting, ranking and clustering grades.

| Rank (s) | ID ($k$) | Model | Brand | Release Date | Price | Fuzzy Item Value | DIV | Grades | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | ($r_{k,12}$) | ($r_{k,17}$) | ($r_{k,1}$) | ($\widehat{X_k}$) | ($\alpha_k$) | Low | Fair | High |
| 1 | 9 | Galaxy E7 | SAMSUNG | Feb–15 | 2898 | (0.579,0.737,0.924) | 0.747 | | | H |
| 2 | 15 | Desire 820 | HTC | Nov–14 | 2998 | (0.547,0.7,0.872) | 0.706 | | | H |
| 3 | 8 | Galaxy A5 | SAMSUNG | Dec–14 | 3298 | (0.531,0.681,0.859) | 0.69 | | | H |
| 4 | 11 | G3 Beat | LG | Aug–14 | 2598 | (0.523,0.671,0.849) | 0.681 | | | H |
| 5 | 18 | Xperia C3 | SONY | Aug–14 | 2698 | (0.527,0.67,0.837) | 0.678 | | | H |
| 6 | 20 | Xperia Z3 Compact | SONY | Sep–14 | 3598 | (0.5,0.647,0.805) | 0.651 | | | H |
| 7 | 16 | Desire Eye | HTC | Nov–14 | 4298 | (0.493,0.637,0.795) | 0.642 | | F | |
| 8 | 7 | GALAXY Note Edge | SAMSUNG | Nov–14 | 7498 | (0.462,0.615,0.782) | 0.62 | | F | |
| 9 | 19 | Xperia Z3 | SONY | Sep–14 | 4998 | (0.468,0.613,0.764) | 0.615 | | F | |
| 10 | 6 | GALAXY Note 4 | SAMSUNG | Sep–14 | 5998 | (0.454,0.608,0.774) | 0.612 | | F | |
| 11 | 12 | G3 | LG | Sep–14 | 5998 | (0.455,0.606,0.768) | 0.61 | | F | |
| 12 | 4 | iPhone 6 Plus | Apple | Sep–14 | 6388 | (0.43,0.573,0.747) | 0.583 | | F | |
| 13 | 14 | Butterfly 2 | HTC | Sep–14 | 5498 | (0.441,0.579,0.729) | 0.583 | | F | |
| 14 | 10 | GALAXY S5 LTE | SAMSUNG | Apr–14 | 4498 | (0.422,0.575,0.742) | 0.58 | | F | |
| 15 | 3 | iPhone 6–16 GB | Apple | Sep–14 | 5588 | (0.424,0.56,0.73) | 0.571 | | F | |
| 16 | 17 | ONE M8 | HTC | Apr–14 | 5598 | (0.374,0.505,0.648) | 0.509 | L | | |
| 17 | 5 | GALAXY Note 3 | SAMSUNG | Sep–13 | 4398 | (0.347,0.492,0.658) | 0.499 | L | | |
| 18 | 13 | G2–32 GB | LG | Sep–13 | 3998 | (0.345,0.479,0.641) | 0.488 | L | | |
| 19 | 1 | iPhone 5S–16 GB | Apple | Sep–13 | 4688 | (0.311,0.445,0.614) | 0.457 | L | | |
| 20 | 2 | iPhone 5S–32 GB | Apple | Sep–13 | 5088 | (0.306,0.442,0.611) | 0.453 | L | | |

example. From Table 3, $\widehat{c_6} = \hat{\beta}_{3,4,1}$. Next, $H(\widehat{c_6}) = 4$, $\widehat{c}_6^4 = \hat{\beta}_{3,4,1}, \widehat{c}_6^3 = \hat{\beta}_{3,4}$, $\widehat{c}_6^2 = \hat{\beta}_3$ and $\widehat{c}_6^1 = \hat{\beta}$. Therefore,

$$\widehat{\lambda_6} = \lambda(\hat{\beta}_{3,4,1}) = \textstyle\prod_{h=H(\widehat{\lambda_6})=4}^{2} \lambda(\widehat{c}_6^h | \widehat{c}_6^{h-1})$$
$$= \lambda(\widehat{c}_6^4|\widehat{c}_6^3)\cdot\lambda(\widehat{c}_6^3|\widehat{c}_6^2)\cdot\lambda(\widehat{c}_6^2|\widehat{c}_6^1)$$
$$= \lambda(\hat{\beta}_{3,4,1}|\hat{\beta}_{3,4})\cdot\lambda(\hat{\beta}_{3,4}|\hat{\beta}_3)\cdot\lambda(\hat{\beta}_3|\hat{\beta})$$
$$= (0.358, 0.42, 0.481)\cdot(0.116, 0.132, 0.147)\cdot(0.299, 0.319, 0.34)$$
$$= (0.012, 0.018, 0.024)$$

After the data are normalized and the fuzzy weights for the measurable features are calculated, the FIVs can be derived by the Fuzzy Arithmetic Mean (Eq. (12)), and the results are presented in Table 8. According to the user preference, Galaxy E7 is the best choice.

### 3.6. Items sorting and ranking

As a Fuzzy Item Value is a fuzzy triangular number and cannot be directly determined for the items sorting and ranking, a Defuzzified Item Value (*DIV*), a crisp value, is derived from *FIV* by using the centroid defuzzification operator (Eq. (13)), and thus can be determined. With respect to the *DIV*s, the items are sorted and ranked from the highest value to the lowest value by using Algorithm 1 implementing the general forms defined in Eqs. (14)-(15) . The results of *DIV*, sorting and ranking results for each item are presented in Table 8.

### 3.7. Fuzzy grade clustering

The detailed computation steps of the *FGC* algorithm are presented in Section 2.7. Either sorted items$\{\widehat{X_k}\}$or non-sorted items $\{\theta_s\}$ can be computed by *FGC*. The cluster patterns and fuzzy centers of initial and

**Table 9**
Fuzzy centers, cluster patterns and grade clusters.

| Preference grade | Initial Fuzzy center | Final fuzzy center | Clusters with $\{\widehat{X}_k\}$ | Clusters with $\{\theta_s\}$ |
|---|---|---|---|---|
| Low | (0.352,0.491,0.664) | (0.337,0.473,0.634) | {1,2,5,13,17} | {17,5,13,1,2} |
| Fair | (0.443,0.59,0.768) | (0.450,0.596,0.759) | {3,4,6,7,10,12,14,16,19} | {16,7,19,6,12,4,14,10,3} |
| High | (0.534,0.688,0.872) | (0.535,0.684,0.858) | {8,9,11,15,18.,20} | {9,15,8,11,18,20} |

**Table 10**
User B's pairwise comparisons to select smartphones.

| $\widehat{B}_0$ | $\widehat{\beta}_1$ | $\widehat{\beta}_2$ | $\widehat{\beta}_3$ | $\widehat{\beta}_4$ |
|---|---|---|---|---|
| $\widehat{\beta}_1$ | $\bar{0}$ | $\bar{8}^-$ | $\bar{1}^-$ | $\bar{2}^-$ |
| $\widehat{\beta}_2$ | $\bar{8}^+$ | $\bar{0}$ | $\bar{8}^+$ | $\bar{7}^+$ |
| $\widehat{\beta}_3$ | $\bar{1}^+$ | $\bar{8}^-$ | $\bar{0}$ | $\bar{1}^-$ |
| $\widehat{\beta}_4$ | $\bar{2}^+$ | $\bar{7}^-$ | $\bar{1}^+$ | $\bar{0}$ |
| $\widehat{AI}$ =0.049 | | | | |

| $\widehat{B}_8$ | Apple | Samsung | LG | HTC | Sony |
|---|---|---|---|---|---|
| Apple | $\bar{0}$ | $\bar{8}^+$ | $\bar{8}^+$ | $\bar{8}^+$ | $\bar{8}^+$ |
| Samsung | $\bar{8}^-$ | $\bar{0}$ | $\bar{1}^+$ | $\bar{2}^+$ | $\bar{1}^+$ |
| LG | $\bar{8}^-$ | $\bar{1}^-$ | $\bar{0}$ | $\bar{1}^+$ | $\bar{1}^+$ |
| HTC | $\bar{8}^-$ | $\bar{2}^-$ | $\bar{1}^-$ | $\bar{0}$ | $\bar{1}^-$ |
| Sony | $\bar{8}^-$ | $\bar{1}^-$ | $\bar{1}^-$ | $\bar{1}^+$ | $\bar{0}$ |
| $\widehat{AI}$ =0.078 | | | | | |

| $\widehat{B}_9$ | iOS | Android |
|---|---|---|
| iOS | $\bar{0}$ | $\bar{8}^+$ |
| Android | $\bar{8}^-$ | $\bar{0}$ |
| $\widehat{AI}$ =0 | | |

final steps with respect to $\{\widehat{X}_k\}$ and $\{\theta_s\}$are presented in Table 9. The clusters have the same items with respect to the inputs of $\{\widehat{X}_k\}$ and $\{\theta_s\}$, but the items are listed in different positions in the same clusters. The sorted items are recommended to be used for *FGC* since the items are ordered with respect to *DIV*s, and sorted items lead to faster clustering convergence. When $G$=3, the items are clustered into three grades according to user preferences and feature values. According to the results presented in Table 8, the smartphones with the lower price (less than HKD 4000) and with better device functions are clustered into one group, as the user preference to purchase a smartphone is price sensitive (i.e.$\widehat{\beta}_1$**)** and function-oriented (i.e. $\widehat{\beta}_3$**)** according to the overall evaluations demonstrated in Figs. 2–4 and Table 5.

### 3.8. Item recommendation

Recommendation can be implemented according to the item patterns calculated which are presented in Table 9. For example, if the user searches product item 5, the system will recommend the rest items in the same cluster of the searched item, i.e. items 17, 13, 1 and 2, to the user. The recommendation order is based on the DIVs of the items in the same cluster whilst the unrelated items in the other clusters are excluded. If the number of options is higher, the number of grading clusters should be accordingly increased. The system can therefore recommend the much customized products to the users according to the computed patterns.

### 3.9. Customization of FCPC-RGC

Another user (User B) fills up the forms which the rating scores are shown in Table 10, whilst rating scores of the previous user mentioned in Section 3 (User A) are presented in Tables 5, 7. The preference results in fuzzy number are presented in Table 11. According to the results in Table 11, User A is price sensitive and looks for better smartphone device with better function, whilst User B has very strong brand loyalty for Apple and keeps with the trends of the products. According to the individual preferences, the system will produce customized ranks and clusters of the products. The results for User B are shown in Table 12, whilst the results for User A are shown in Table 8. Table 8 shows the system returns the high priorities for the products with lower price and better features for User A, whilst Table 12 shows the system returns the high priorities for the products with Apple and latest models for User B.

## 4. Comparisons

Unlike the conventional recommender systems, e.g. collaborative filtering (Goldberg et al., 1992; Herlocker et al., 2004; Shi et al., 2014), content-based filtering (Lops et al., 2011; Pazzani and Billsus, 2007), and hybrid approaches (de Campos et al., 2010; Salter and Antonopoulos, 2006; Pazzani, 1999), the proposed FCPC-RGC can typically handle the customization problems of the individual users for new items such as the trending products, as the proposed method does not rely on the other users' rating history but an individual user's current rating preference which is elicited as the preference weights or scales values for either new or old products (but out-of-date products may not be considered by the consumers).

The proposed application of smartphone recommendation could not be directly reproduced by the other alternative methods such as in (Cao and Li, 2007; Rokach and Kisilevich, 2012; Saaty, 1977, 1980, 2005; Liu and Shih, 2005; Wang et al., 2008; Yuen, 2012b) due to the differences of the data structures and the designs of hybrid systems. The proposed hybrid

**Table 11**
Two users' preferences to select smartphones by FCPC.

| | | User A | User B |
|---|---|---|---|
| Feature weight | Price attractiveness | (0.306,0.354,0.396) | (0.132,0.174,0.222) |
| | Brand | (0.083,0.118,0.16) | (0.368,0.41,0.438) |
| | Smartphone device | (0.299,0.319,0.34) | (0.174,0.194,0.222) |
| | Release date | (0.201,0.208,0.215) | (0.215,0.222,0.229) |
| Brands | Apple | (0.231,0.271,0.311) | (0.302,0.342,0.364) |
| | Samsung | (0.173,0.204,0.236) | (0.151,0.182,0.218) |
| | LG | (0.164,0.187,0.209) | (0.147,0.169,0.196) |
| | HTC | (0.147,0.16,0.173) | (0.133,0.147,0.164) |
| | Sony | (0.173,0.178,0.182) | (0.156,0.16,0.169) |
| OS | iOS | (0.583,0.667,0.75) | (0.639,0.722,0.778) |
| | Android | (0.306,0.333,0.361) | (0.25,0.278,0.333) |

**Table 12**
Results of items sorting, ranking and clustering grades.

| Rank (s) | ID (*k*) | Model | Brand | Price | Release Date | Fuzzy Item Value | DIV | Grades | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | Low | Fair | High |
| 1 | 4 | iPhone 6 Plus | Apple | 6388 | Sep−14 | (0.566,0.722,0.871) | 0.72 | | | H |
| 2 | 3 | iPhone 6−16 GB | Apple | 5588 | Sep−14 | (0.56,0.712,0.859) | 0.71 | | | H |
| 3 | 9 | Galaxy E7 | SAMSUNG | 2898 | Feb−15 | (0.515,0.662,0.841) | 0.673 | | | H |
| 4 | 8 | Galaxy A5 | SAMSUNG | 3298 | Dec−14 | (0.481,0.622,0.794) | 0.632 | | | H |
| 5 | 15 | Desire 820 | HTC | 2998 | Nov−14 | (0.467,0.596,0.751) | 0.605 | | F | |
| 6 | 7 | GALAXY Note Edge | SAMSUNG | 7498 | Nov−14 | (0.453,0.594,0.754) | 0.6 | | F | |
| 7 | 11 | G3 Beat | LG | 2598 | Aug−14 | (0.445,0.577,0.743) | 0.588 | | F | |
| 8 | 6 | GALAXY Note 4 | SAMSUNG | 5998 | Sep−14 | (0.435,0.575,0.737) | 0.582 | | F | |
| 9 | 18 | Xperia C3 | SONY | 2698 | Aug−14 | (0.456,0.57,0.713) | 0.58 | | F | |
| 10 | 1 | iPhone 5S−16 GB | Apple | 4688 | Sep−13 | (0.429,0.578,0.723) | 0.577 | | F | |
| 11 | 20 | Xperia Z3 Compact | SONY | 3598 | Sep−14 | (0.455,0.57,0.706) | 0.577 | | F | |
| 12 | 2 | iPhone 5S−32 GB | Apple | 5088 | Sep−13 | (0.428,0.578,0.723) | 0.576 | | F | |
| 13 | 16 | Desire Eye | HTC | 4298 | Nov−14 | (0.446,0.568,0.71) | 0.575 | | F | |
| 14 | 12 | G3 | LG | 5998 | Sep−14 | (0.432,0.564,0.715) | 0.571 | | F | |
| 15 | 19 | Xperia Z3 | SONY | 4998 | Sep−14 | (0.444,0.557,0.686) | 0.562 | | F | |
| 16 | 14 | Butterfly 2 | HTC | 5498 | Sep−14 | (0.411,0.528,0.663) | 0.534 | | F | |
| 17 | 10 | GALAXY S5 LTE | SAMSUNG | 4498 | Apr−14 | (0.384,0.525,0.688) | 0.532 | | F | |
| 18 | 17 | ONE M8 | HTC | 5598 | Apr−14 | (0.346,0.458,0.586) | 0.463 | L | | |
| 19 | 5 | GALAXY Note 3 | SAMSUNG | 4398 | Sep−13 | (0.304,0.439,0.599) | 0.447 | L | | |
| 20 | 13 | G2−32 GB | LG | 3998 | Sep−13 | (0.298,0.418,0.568) | 0.428 | L | | |

**Table 13**
Cluster results of different clustering methods.

| Clustering method | Cluster results |
|---|---|
| FCPC-RGC (User A's preference) | {9,15,8,11,18,20},{16,7,19,6,12,4,14,10,3}, {17,5,13,1,2} |
| FCPC-RGC (User B preference) | {4,3,9,8},{15,7,11,6,18,1,20,2,16,12,19,14,10}, {17,5,13} |
| k-means | {1,2,3},{4,8,9,11,13,14,15,16,17,18}, {5,6,7,10,12,19,20}; or {1,2,3},{4,5,6,7,10,12,13,14,17,19,20}, {8,9,11,15,16,18 }. |
| Fuzzy c-means | {1,2,3},{4,8,9,11,13,15,16,18}, {5,6,7,10,12,14,17,19,20} |
| Hierarchical Clustering (average or single link) | {1,2,3},{4}, {5,6,7,8,9 10 11 12 13 14 15 16 17 18 19 20} |
| Hierarchical Clustering (complete link) | {1,2,3},{4,8,9,11,14,15,16,17,18}, {5,6,7,10,12,13,19,20} |

approach comprises three parts: Fuzzy Cognitive Pairwise Comparison (FCPC), Fuzzy Ranking (FR) and Fuzzy Grade Clustering (FGC), and therefore the comparisons below are conducted and discussed in the order of clustering, ranking, and rating respectively.

### *4.1. Clustering*

Data in Table A1 are attempted to be analyzed by k-means, fuzzy c-means, and Hierarchical clustering on the basis of the packages of *R* language, but there are several shortages for these clustering methods. Firstly, they cannot directly handle such non-numeric data such as the linguistic labels for the features of Brands and OS. Secondly, the cluster results are nominal and not in order. Thirdly, some clustering methods such as *k*-means could produce different cluster results by repeatedly running several times due to random local searches. Fourthly, they cannot handle the fuzzy data. Finally, as the clustering methods above do not consider the weights for features as input, it is not incorporated with user preferences to produce customized patterns.

To handle the labels values for the use of the clustering methods above for comparisons, one of the conventional approaches is to assign the integer values to the nominal labels. For the Brands feature, the labels, Apple, HTC, LG, Samsung, and SONY, are assigned by 1–5. For the OS feature, Android and iOS are assigned by 1 and 2. For the max video resolution feature, 1 is assigned to 1080p whilst 4 is assigned to 4k. For the release date feature, Sep-13 is the start point and assigned by 1. The data are standardized to set off the variances problem of the ranges of the different features values. The cluster results are presented in Table 13.

Table 13 shows that different cluster methods produce different cluster results. When running the same k-means programme for several times, the clustering results can be different due to its random local search leading to the convergence. The Hierarchical Clustering with the same setting can produce the same result for each running, but the choice of link for hierarchical is still uncertain and requires further testing. The scope of this paper is not to discuss metrics for their clustering results as many papers already discussed and normally concluded that no typical method outperforms the others.

The proposed FCPC-RGC could offset the shortages mentioned above. The proposed method can process labels values into meaningful numeric values with respect to the customers' preferences, and ultimately the customized patterns are produced due to consideration of the user references. The conventional approach presented above for setting the labels values into integer is quite arbitrary as the sequence of the integer values being assigned to a set of the categorical labels in different orders very likely leads to different clustering results. Since the FCPC-RGC uses the statistical method to define the initial center values after aggregation of the dimensions in previous stage, FCPC-RGC should produce the same result whatever how many times the programme repeatedly runs.

Regarding the complexity of the Fuzzy Grade Clustering, Step i takes *O(GM)*; Step ii takes *O(GM)*, Step iii takes *O(GM)*; Step iv takes *O(G)*; After the *R* iterations in Step v, the complexity of the FGC is *O(R(GM+G))*. The number of clusters, number of dimensions and number of iterations could be omitted for the time complexity of Big O notation as they should be relative small. Only the number of items, *M*, are considered and data are organized in matrix form. Finally, the complexity of *FGC* is *O(M)*, whilst the time complexity for the k-means and the fuzzy c-means are *O(M)* and the time complexity for the hierarchical clustering is $O(M^3)$, which is the lowest among these methods.

### *4.2. Ranking*

The similar research is rare and one similar study is identified. Işiklar and Büyüközkan (2007) applied AHP and TOPSIS to evaluate mobile phones. As this study uses fuzzy number, methods in (Işiklar and Büyüközkan, 2007) cannot be directly applied to the proposed application. The comparisons

**Table 14**
Aggregation Values and Ranks for different fusion method.

| Fusion method | Results | |
|---|---|---|
| Fuzzy Arithmetic Mean | Aggregation Values | {0.747,0.706,0.690,0.681,0.678,0.651,0.642,0.620,0.615,0.612,0.610,0.583,0.583,0.580,0.571,0.509,0.499,0.488,0.457,0.453} |
| | Rank | {9,15,8,11,18,20,16,7,19,6,12,4,14,10,3,17,5,13,1,2} |
| Fuzzy TOPSIS (Chen (Chen, 2000)) | Aggregation Values | {0.0280,0.0278,0.0346,0.0353,0.0304,0.0370,0.0374,0.0415,0.0449,0.0351,0.0409,0.0368,0.0297,0.035-1,0.0424,0.0386,0.0307,0.0406,0.0370,0.0391} |
| | Rank | {9,15,8,11,18,20,16,7,6,19,12,4,14,10,3,17,5,13,1,2} |
| Fuzzy TOPSIS (Yuen (Yuen, 2014c)) | Aggregation Values | {0.249,0.244,0.426,0.445,0.313,0.488,0.500,0.608,0.695,0.438,0.591,0.483,0.294,0.439,0.630,0.529,0.324,0.584,0.488,0.543} |
| | Rank | {9,15,8,11,18,20,16,7,6,19,12,4,14,10,3,17,5,13,1,2} |

between fuzzy TOPSIS and the proposed Fuzzy Arithmetic Mean (FAM) are conducted and discussed. According to the literature, the earliest design for fuzzy TOPSIS was proposed by Chen (Chen, 2000), which was further modified by Yuen (Yuen, 2014c). The major differences between Chen's and Yuen's methods are the definitions of fuzzy positive-ideal solution (FPIS) and fuzzy negative-ideal solution (FNIS). After FPIS and FNIS are identified, the distance measures for FPIS and FNIS are calculated, and relative measures are further computed. These three steps could be regarded as fuzzy aggregation and could simply be implemented by the Fuzzy Arithmetic Mean (FAM) (Eq. (12)) in this study.

Table 14 presents the aggregation values and rank with different fusion methods. According to the aggregated values, both Fuzzy TOPSIS methods produce the same rank which is slightly different from FAM, especially for the items in ranks 6 and 19. Values by Chen's method (Chen, 2000) are very small and the relative differences among adjacent items are narrow. Yuen (Yuen, 2014c) proposed the improvement with the better aggregation values. In this application, all features are the positive criteria for the fuzzy TOPSIS due to the normalization step for the data preparation. Although many fuzzy aggregation operators are proposed, the Fuzzy Arithmetic Mean (FAM) is chosen in this paper since it is simple and easy to understand by the others, whilst there are no good benchmarking methods for the choice of aggregation operator.

The core difference between fuzzy TOPSIS and the proposed FAM is the fuzzy aggregation part, whilst the other steps are similar. For the complexity to aggregate a fuzzy data table of $M$ item sets of $N$ measureable features, the FAM takes $O(NM)$. For these three steps for aggregation in fuzzy TOPSIS, Yuen's method (Yuen, 2014c) takes $O(4NM+M)$, whilst Chen's method (Chen, 2000) takes $O(2NM+M)$. *FAM* runs faster.

### 4.3. Rating

User preferences for the product data are typically essential for the customized results. Rating is the essential method to elicit the user preferences. The direct rating method is the popular one, but has several drawbacks. An example is shown in Fig. 5. After a score is chosen for a question, this score can be used in crisp or fuzzy value. As a consumer usually is greedy and wants a product with the famous brand, best product attributes, latest model and very good price, and ultimately he or she very likely chooses the maximum scores for all factors. Comparing with the direct rating method, the pairwise comparison rating shown in Fig. 2 is the ideal method to compare the relative importance of pairs of items.

The first use of Pairwise Comparisons (PC) is often attributed to Ramon Llull, the 13th-century mystic and philosopher (Koczkodaj et al., 2016). The study (Saaty, 1977) had a considerable impact on PC research and led to the Analytic Hierarchy Process (AHP) becoming a proprietary eponym for PC (Koczkodaj et al., 2016). Whilst there are a lot of debates for the validity of the AHP, many authors of the AHP application papers seem not to be aware of the problems of AHP in various aspects. With respect to the rating scale problem of AHP, the Cognitive Network Process (CNP) (Yuen, 2009, 2014a, 2012a, 2014b) is proposed as the ideal alternative of the Analytic Hierarchy Process (AHP). The numerical definition of the AHP's paired ratio scale inappropriately represents the human intuitive judgement of paired difference, and thus CNP uses paired interval scale to replace paired ratio scale. The comparison is called the Cognitive Pairwise Comparison.

Whilst the fuzzy number is applied to the PC, there are FAHP and FCPC. Whilst there are several types of FAHP, this paper chooses the FAHP version in (Wang et al., 2008; Wang and Chin, 2008), which is considered to be more reliable than the other types such as the Extent Analysis Method (EAM) (Chang, 1996). To produce the results by using FAHP, the rating scale should be converted. Conversion of rating scale between FAHP and FCPC was presented Table I in (Yuen, 2014a). Table 15 shows the results of ranks, clusters and DIVs for FAHP and FCPC respectively. As FCPC and FAHP produce different values of weights, DIV, rank and cluster values are different. The details for the discussion of FCPC and FAHP can be found in (Yuen, 2009, 2014a, 2012a, 2014b).

There are three advantages of Fuzzy Cognitive Pairwise Comparison. The first one is its rating interface (e.g. Fig. 2). Unlike the classical direct rating method (Fig. 5), FCPC can show that rating score represents which one is compared to which one, and FCPC can identify the unreasonable ratings by checking the FAI scores. The second advantage is that the rating scores can be used to evaluate the weights for the features or alternatives with respective to the users' preferences. The third advantage is that the FCPC can convert the linguistic labels into fuzzy numbers with respect to the users' preferences, which can be used for clustering algorithms, as the linguistic labels cannot be directly used for many clustering algorithms. For example, with respect to the results derived from rating matrices by using FCPC shown in Tables 5, 7, linguistic labels of Brands and OS in Table A1 will be converted into fuzzy numbers shown in Table A2.

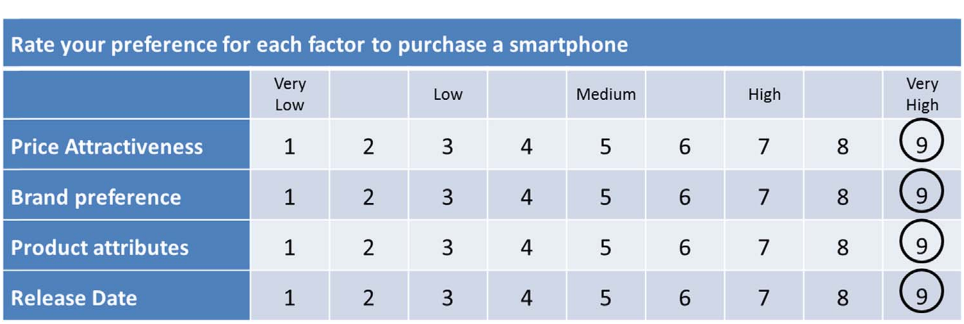


**Fig. 5.** A form for the customer preference for each factor to buy a smartphone using direct rating score.

**Table 15**
Results for User A's case by using FCPC and FAHP respectively.

| Rating methods | Results | |
|---|---|---|
| FCPC | Rank and cluster | {9,15,8,11,18,20},{16,7,19,6,12,4,14,10,3},{17,5,13,1,2} |
| | DIV | {0.72,0.71,0.673,0.632,0.605,0.6},{0.588,0.582,0.58,0.577,0.577,0.576,0.575,0.571,0.562},{0.534,0.532,0.463,0.447,0.428} |
| FAHP | Rank and cluster | {9,15,18,11,8,20},{16,19,10,6,12,14,5},{7,13,4,17,3,1,2} |
| | DIV | {0.737,0.737,0.727,0.723,0.673,0.661},{0.607,0.586,0.581,0.547,0.546,0.543,0.531},{0.523,0.522,0.485,0.482,0.479,0.450,0.437} |

## 5. Discussions

### 5.1. Usability of FCPC-RGC

Many smartphone purchasing websites just provide basic filtering and sorting functions to search smartphones. For the filtering functions, the users normally select the filtering criteria of product features and the systems only return the products meeting the users' selection criteria with omitting the other features. For the sorting functions, the systems normally return the sorted items by selected feature such as price. Therefore, such filtering and sorting functions do not consider aggregation results of user preferences for the overall features with respect to all products, and may not produce the customized results. FCPC-RGC addresses these issues by considering all features which preferences are expressed by users.

Two input sources of the proposed FCPC-RGC are used: product feature data and user preferences. Once the product engineers initially set up the product features and parameters, the feature data can be obtained from the external sources such as the retailors. After a user fills up a form to express his/her preference, and the system will automatically produce product recommendation results in proper ranks and clusters by using the proposed FCPC-RGC.

To further illustrate the usability in user perspectives for the proposed methods, Section 3 demonstrates how the proposed methods are applied to the smartphone product recommendation. Product feature data from external sources with respect to the feature specification shown in Fig. 3 are presented in Table A1. The domain experts can design pairwise comparison matrices, which are shown in Table 6 as default values, to evaluate the levels of features. A user needs to fill a form shown in Fig. 2 to express his/her relative purchasing preference, which data are presented in Table 5. Similarly, the user is asked to fill the forms about his/her preference about Brand and OS. The rating results are saved in the system and presented in Table 7. For the quick search for a better recommendation results, a user only needs to fill up three forms to express his/her preference. The system also includes the search function with advance mode to capture better user's views on the features to replace the default settings shown in Table 6. After the user submits the form, the proposed system will calculate the product values and grade them into different groups to present the products information in proper order, e.g. Table 8. Section 3.9 shows the customization advantages with respect to different users' preferences.

### 5.2. Scalability of FCPC-RGC

The proposed FCPC-RGC can address the scalability of the features, product data, and users. Regarding the scalability of users, the experiment in Section 3.9 shows preferences inputs and recommendation results of User A are independent of User B. The system can serve different users with individual search requirements. Regarding the scalability of product features, the product specification is organized as a tree structure. The domain experts can define different numbers, sizes and levels of branches for the product tree. Only data for the measurable feature indicators($c_1, \ldots, c_k, \ldots, c_N$) located in the external nodes of the tree are fetched. For example, Table 3 shows how measurable feature indicators $\{c_k\}$ are mapped with the external nodes of the tree ($\hat{\beta}$). The domain experts can define reasonable $N$ features.

Regarding scalability of product data $\{r_{kj}\}$, the number of items, $M$, is independent of measurable feature indicators, $N$. Once the weights are derived by Eqs. (7)–(8) from POMs on the basis of the tree specification, the weights can be calculated by Eq. (11). The indices for the variables in the equations are not fixed. The system therefore can calculate either a few or many items, e.g. the size of $M$. Any new item data is fetched, the system will recalculate and update the recommendation results.

## 6. Conclusions

The high-end consumer products rapidly change over time. The old model data may not be useful to create the patterns for the latest product recommendation. Consumer perceived values for the product feature change over time. In the competitive market, the manufacturers offer diverse products to target different market segments. Consumers may face challenges to find their preferred products. To address the above challenges, this paper proposes the Fuzzy Cognitive Pairwise Comparison for Ranking and Grade Clustering (FCPC-RGC) comprised of three major parts of eight steps: Fuzzy Cognitive Pairwise Comparison (FCPC), Fuzzy Ranking (FR) and Fuzzy Grade Clustering (FGC), which the advantages are discussed in Sections 4 and 5. FCPC is used to elicit the customers' perceived preferences for the features and convert the nominal labels to fuzzy numbers. FR is used to sort and rank the items. FGC is used to cluster items into ordinal grades according to the users' preferences and item features. An application to smartphone recommendation using the proposed approach is demonstrated for the usability and validity.

The proposed method can typically handle the recommender problems of trending products due to its scalability to proceed the updates of product features and items and usability to capture the users' preferences. On the basis of the current established work, extension of the study will investigate the recommendation problems with timely responses from the large scale data in a distributed environment. The future study also considers to build hybrid systems with the other machine learning methods such as evolutionary and swarm algorithms to improve the efficiency of computation and accuracy of the recommendation results from the large scale data. With the demonstration of smartphones recommendation, the proposed method can also be applied to the other trending products recommendations such as music, movie, books and consumer electronic products.

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## Appendix

See Tables A1–A5.

**Table A1**
Smartphone Feature Data.

| ID | Model | Price | Brands | OS | Processer | PPI | DisplaySize | PhoneDepth | Weight | ROM | RAM | MES | Pri Cam. | Front Camera | Max Video Res. | Bat Cap | MaxTalkTime | Release Date |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | iPhone 5S–16 GB | 4688 | Apple | iOS | 2556 | 326 | 4 | 7.6 | 112 | 16 | 1 | 0 | 8 | 1.2 | 1080p | 1570 | 10 | Sep–13 |
| 2 | iPhone 5S–32 GB | 5088 | Apple | iOS | 2556 | 326 | 4 | 7.6 | 112 | 32 | 1 | 0 | 8 | 1.2 | 1080p | 1570 | 10 | Sep–13 |
| 3 | iPhone 6–16 GB | 5588 | Apple | iOS | 2794 | 326 | 4.7 | 6.9 | 129 | 16 | 1 | 0 | 8 | 1.2 | 1080p | 1810 | 14 | Sep–14 |
| 4 | iPhone 6 Plus | 6388 | Apple | iOS | 2917 | 401 | 5.5 | 7.1 | 172 | 16 | 1 | 0 | 8 | 1.2 | 1080p | 2915 | 24 | Sep–14 |
| 5 | GALAXY Note 3 | 4398 | Samsung | Android | 2982 | 386 | 5.7 | 8.38 | 168 | 16 | 3 | 64 | 13 | 2.1 | 4k | 3200 | 21 | Sep–13 |
| 6 | GALAXY Note 4 | 5998 | Samsung | Android | 3272 | 515 | 5.7 | 8.5 | 176 | 16 | 3 | 128 | 16 | 3.7 | 4k | 3220 | 20 | Sep–14 |
| 7 | GALAXY Note Edge | 7498 | Samsung | Android | 3302 | 524 | 5.6 | 8.3 | 174 | 32 | 3 | 128 | 16 | 3.7 | 4k | 3000 | 18 | Nov–14 |
| 8 | Galaxy A5 | 3298 | Samsung | Android | 1426 | 294 | 5 | 6.7 | 123 | 16 | 2 | 64 | 13 | 5 | 1080p | 2300 | 17 | Dec–14 |
| 9 | Galaxy E7 | 2898 | Samsung | Android | 1384 | 267 | 5.5 | 7.3 | 141 | 16 | 2 | 64 | 13 | 5 | 1080p | 2950 | 16 | Feb–15 |
| 10 | GALAXY S5 LTE | 4498 | Samsung | Android | 2869 | 431 | 5.1 | 8.1 | 145 | 16 | 2 | 128 | 16 | 2.1 | 4k | 2800 | 21 | Apr–14 |
| 11 | G3 Beat | 2598 | LG | Android | 1121 | 294 | 5 | 10.3 | 133 | 8 | 1 | 32 | 8 | 1.3 | 1080p | 2540 | 15 | Aug–14 |
| 12 | G3 | 5998 | LG | Android | 2465 | 534 | 5.5 | 9.3 | 152 | 32 | 3 | 128 | 13 | 2.1 | 4k | 3000 | 21 | Sep–14 |
| 13 | G2–32 GB | 3998 | LG | Android | 2221 | 424 | 5.2 | 9.1 | 143 | 32 | 2 | 0 | 13 | 2.1 | 1080p | 3000 | 17 | Sep–13 |
| 14 | Butterfly 2 | 5498 | HTC | Android | 3013 | 440 | 5 | 9.99 | 151 | 16 | 2 | 128 | 13 | 5 | 1080p | 2700 | 24 | Sep–14 |
| 15 | Desire 820 | 2998 | HTC | Android | 2515 | 267 | 5.5 | 7.74 | 155 | 16 | 2 | 128 | 13 | 8 | 1080p | 2600 | 23 | Nov–14 |
| 16 | Desire Eye | 4298 | HTC | Android | 2861 | 424 | 5.2 | 8.5 | 154 | 16 | 2 | 128 | 13 | 13 | 1080p | 2400 | 20 | Nov–14 |
| 17 | ONE M8 | 5598 | HTC | Android | 2761 | 440 | 5 | 9.35 | 160 | 16 | 2 | 128 | 4 | 5 | 1080p | 2600 | 20 | Apr–14 |
| 18 | Xperia C3 | 2698 | SONY | Android | 1181 | 267 | 5.5 | 7.6 | 150 | 8 | 1 | 32 | 8 | 5 | 1080p | 2500 | 25 | Aug–14 |
| 19 | Xperia Z3 | 4998 | SONY | Android | 2805 | 424 | 5.2 | 7.4 | 153 | 16 | 3 | 128 | 20.7 | 2.2 | 4k | 3100 | 19 | Sep–14 |
| 20 | Xperia Z3 Compact | 3598 | SONY | Android | 2800 | 319 | 4.6 | 8.64 | 126 | 16 | 2 | 128 | 20.7 | 2.2 | 4k | 2600 | 14 | Sep–14 |

**Table A2**
Fuzzy data I for smartphone measurable features.

| ID | Model | Price | Brands | OS | Processer | PPI | DisplaySize | PhoneDepth | Weight | ROM |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | iPhone 5S–16 GB | (4219,4688,4922) | (0.231,0.271,0.311) | (0.583,0.667,0.75) | (4219,4688,5157) | (293,326,342) | (3.8,4,4.2) | (7.4,7.6,7.8) | (110,112,117) | (14.4,16,16.8) |
| 2 | iPhone 5S–32 GB | (4579,5088,5342) | (0.231,0.271,0.311) | (0.583,0.667,0.75) | (4579,5088,5597) | (293,326,342) | (3.8,4,4.2) | (7.4,7.6,7.8) | (110,112,117) | (28.8,32,33.6) |
| 3 | iPhone 6–16 GB | (5029,5588,5867) | (0.231,0.271,0.311) | (0.583,0.667,0.75) | (5029,5588,6147) | (293,326,342) | (4.5,4.7,4.9) | (6.7,6.9,7.1) | (127,129,134) | (14.4,16,16.8) |
| 4 | iPhone 6 Plus | (5749,6388,6707) | (0.231,0.271,0.311) | (0.583,0.667,0.75) | (5749,6388,7027) | (361,401,421) | (5.3,5.5,5.7) | (6.9,7.1,7.3) | (170,172,177) | (14.4,16,16.8) |
| 5 | GALAXY Note 3 | (3958,4398,4618) | (0.173,0.204,0.236) | (0.306,0.333,0.361) | (3958,4398,4838) | (347,386,405) | (5.5,5.7,5.9) | (8.18,8.38,8.58) | (166,168,173) | (14.4,16,16.8) |
| 6 | GALAXY Note 4 | (5398,5998,6298) | (0.173,0.204,0.236) | (0.306,0.333,0.361) | (5398,5998,6598) | (464,515,541) | (5.5,5.7,5.9) | (8.3,8.5,8.7) | (174,176,181) | (14.4,16,16.8) |
| 7 | GALAXY Note Edge | (6748,7498,7873) | (0.173,0.204,0.236) | (0.306,0.333,0.361) | (6748,7498,8248) | (472,524,550) | (5.4,5.6,5.8) | (8.1,8.3,8.5) | (172,174,179) | (28.8,32,33.6) |
| 8 | Galaxy A5 | (2968,3298,3463) | (0.173,0.204,0.236) | (0.306,0.333,0.361) | (2968,3298,3628) | (265,294,309) | (4.8,5,5.2) | (6.5,6.7,6.9) | (121,123,128) | (14.4,16,16.8) |
| 9 | Galaxy E7 | (2608,2898,3043) | (0.173,0.204,0.236) | (0.306,0.333,0.361) | (2608,2898,3188) | (240,267,280) | (5.3,5.5,5.7) | (7.1,7.3,7.5) | (139,141,146) | (14.4,16,16.8) |
| 10 | GALAXY S5 LTE | (4048,4498,4723) | (0.173,0.204,0.236) | (0.306,0.333,0.361) | (4048,4498,4948) | (388,431,453) | (4.9,5.1,5.3) | (7.9,8.1,8.3) | (143,145,150) | (14.4,16,16.8) |
| 11 | G3 Beat | (2338,2598,2728) | (0.164,0.187,0.209) | (0.306,0.333,0.361) | (2338,2598,2858) | (265,294,309) | (4.8,5,5.2) | (10.1,10.3,10.5) | (131,133,138) | (7.2,8,8.4) |
| 12 | G3 | (5398,5998,6298) | (0.164,0.187,0.209) | (0.306,0.333,0.361) | (5398,5998,6598) | (481,534,561) | (5.3,5.5,5.7) | (9.1,9.3,9.5) | (150,152,157) | (28.8,32,33.6) |
| 13 | G2–32 GB | (3598,3998,4198) | (0.164,0.187,0.209) | (0.306,0.333,0.361) | (3598,3998,4398) | (382,424,445) | (5,5.2,5.4) | (8.9,9.1,9.3) | (141,143,148) | (28.8,32,33.6) |
| 14 | Butterfly 2 | (4948,5498,5773) | (0.147,0.16,0.173) | (0.306,0.333,0.361) | (4948,5498,6048) | (396,440,462) | (4.8,5,5.2) | (9.79,9.99,10.19) | (149,151,156) | (14.4,16,16.8) |
| 15 | Desire 820 | (2698,2998,3148) | (0.147,0.16,0.173) | (0.306,0.333,0.361) | (2698,2998,3298) | (240,267,280) | (5.3,5.5,5.7) | (7.54,7.74,7.94) | (153,155,160) | (14.4,16,16.8) |
| 16 | Desire Eye | (3868,4298,4513) | (0.147,0.16,0.173) | (0.306,0.333,0.361) | (3868,4298,4728) | (382,424,445) | (5,5.2,5.4) | (8.3,8.5,8.7) | (152,154,159) | (14.4,16,16.8) |
| 17 | ONE M8 | (5038,5598,5878) | (0.147,0.16,0.173) | (0.306,0.333,0.361) | (5038,5598,6158) | (396,440,462) | (4.8,5,5.2) | (9.15,9.35,9.55) | (158,160,165) | (14.4,16,16.8) |
| 18 | Xperia C3 | (2428,2698,2833) | (0.173,0.178,0.182) | (0.306,0.333,0.361) | (2428,2698,2968) | (240,267,280) | (5.3,5.5,5.7) | (7.4,7.6,7.8) | (148,150,155) | (7.2,8,8.4) |
| 19 | Xperia Z3 | (4498,4998,5248) | (0.173,0.178,0.182) | (0.306,0.333,0.361) | (4498,4998,5498) | (382,424,445) | (5,5.2,5.4) | (7.2,7.4,7.6) | (151,153,158) | (14.4,16,16.8) |
| 20 | Xperia Z3 Compact | (3238,3598,3778) | (0.173,0.178,0.182) | (0.306,0.333,0.361) | (3238,3598,3958) | (287,319,335) | (4.4,4.6,4.8) | (8.44,8.64,8.84) | (124,126,131) | (14.4,16,16.8) |

**Table A3**
Fuzzy Data II for smartphone measurable features.

| ID | RAM | Max. External Storage | Primary Camera | Front Camera | Max Video Res. | Bat Cap | MaxTalkTime | Release Date |
|---|---|---|---|---|---|---|---|---|
| 1 | (0.9,1,1.05) | (0,0,0) | (7.2,8,8) | (1.08,1.2,1.2) | (0.2,0.25,0.3) | (1413,1570,1648) | (9,10,10) | (0,1,2) |
| 2 | (0.9,1,1.05) | (0,0,0) | (7.2,8,8) | (1.08,1.2,1.2) | (0.2,0.25,0.3) | (1413,1570,1648) | (9,10,10) | (0,1,2) |
| 3 | (0.9,1,1.05) | (0,0,0) | (7.2,8,8) | (1.08,1.2,1.2) | (0.2,0.25,0.3) | (1629,1810,1900) | (12.6,14,14) | (12,13,14) |
| 4 | (0.9,1,1.05) | (0,0,0) | (7.2,8,8) | (1.08,1.2,1.2) | (0.2,0.25,0.3) | (2624,2915,3061) | (21.6,24,24) | (12,13,14) |
| 5 | (2.7,3,3.15) | (32,64,64) | (11.7,13,13) | (1.89,2.1,2.1) | (0.9,0.95,1) | (2880,3200,3360) | (18.9,21,21) | (0,1,2) |
| 6 | (2.7,3,3.15) | (64,128,128) | (14.4,16,16) | (3.33,3.7,3.7) | (0.9,0.95,1) | (2898,3220,3381) | (18,20,20) | (12,13,14) |
| 7 | (2.7,3,3.15) | (64,128,128) | (14.4,16,16) | (3.33,3.7,3.7) | (0.9,0.95,1) | (2700,3000,3150) | (16.2,18,18) | (14,15,16) |
| 8 | (1.8,2,2.1) | (32,64,64) | (11.7,13,13) | (4.5,5,5) | (0.2,0.25,0.3) | (2070,2300,2415) | (15.3,17,17) | (15,16,17) |
| 9 | (1.8,2,2.1) | (32,64,64) | (11.7,13,13) | (4.5,5,5) | (0.2,0.25,0.3) | (2655,2950,3098) | (14.4,16,16) | (17,18,19) |
| 10 | (1.8,2,2.1) | (64,128,128) | (14.4,16,16) | (1.89,2.1,2.1) | (0.9,0.95,1) | (2520,2800,2940) | (18.9,21,21) | (7,8,9) |
| 11 | (0.9,1,1.05) | (16,32,32) | (7.2,8,8) | (1.17,1.3,1.3) | (0.2,0.25,0.3) | (2286,2540,2667) | (13.5,15,15) | (11,12,13) |
| 12 | (2.7,3,3.15) | (64,128,128) | (11.7,13,13) | (1.89,2.1,2.1) | (0.9,0.95,1) | (2700,3000,3150) | (18.9,21,21) | (12,13,14) |
| 13 | (1.8,2,2.1) | (0,0,0) | (11.7,13,13) | (1.89,2.1,2.1) | (0.2,0.25,0.3) | (2700,3000,3150) | (15.3,17,17) | (0,1,2) |
| 14 | (1.8,2,2.1) | (64,128,128) | (11.7,13,13) | (4.5,5,5) | (0.2,0.25,0.3) | (2430,2700,2835) | (21.6,24,24) | (12,13,14) |
| 15 | (1.8,2,2.1) | (64,128,128) | (11.7,13,13) | (7.2,8,8) | (0.2,0.25,0.3) | (2340,2600,2730) | (20.7,23,23) | (14,15,16) |
| 16 | (1.8,2,2.1) | (64,128,128) | (11.7,13,13) | (11.7,13,13) | (0.2,0.25,0.3) | (2160,2400,2520) | (18,20,20) | (14,15,16) |
| 17 | (1.8,2,2.1) | (64,128,128) | (3.6,4,4) | (4.5,5,5) | (0.2,0.25,0.3) | (2340,2600,2730) | (18,20,20) | (7,8,9) |
| 18 | (0.9,1,1.05) | (16,32,32) | (7.2,8,8) | (4.5,5,5) | (0.2,0.25,0.3) | (2250,2500,2625) | (22.5,25,25) | (11,12,13) |
| 19 | (2.7,3,3.15) | (64,128,128) | (18.63,20.7,20.7) | (1.98,2.2,2.2) | (0.9,0.95,1) | (2790,3100,3255) | (17.1,19,19) | (12,13,14) |
| 20 | (1.8,2,2.1) | (64,128,128) | (18.63,20.7,20.7) | (1.98,2.2,2.2) | (0.9,0.95,1) | (2340,2600,2730) | (12.6,14,14) | (12,13,14) |

**Table A4**
Normalized Fuzzy data I for smartphone measurable features.

| ID | Price | Brands | OS | Processer | PPI | DisplaySize | PhoneDepth | Weight | ROM |
|---|---|---|---|---|---|---|---|---|---|
| 1 | (0.475,0.499,0.554) | (0.743,0.871,1) | (0.777,0.889,1) | (0.512,0.568,0.625) | (0.522,0.581,0.61) | (0.644,0.678,0.712) | (0.705,0.724,0.743) | (0.94,0.982,1) | (0.429,0.476,0.5) |
| 2 | (0.438,0.46,0.511) | (0.743,0.871,1) | (0.777,0.889,1) | (0.555,0.617,0.679) | (0.522,0.581,0.61) | (0.644,0.678,0.712) | (0.705,0.724,0.743) | (0.94,0.982,1) | (0.857,0.952,1) |
| 3 | (0.399,0.418,0.465) | (0.743,0.871,1) | (0.777,0.889,1) | (0.61,0.677,0.745) | (0.522,0.581,0.61) | (0.763,0.797,0.831) | (0.638,0.657,0.676) | (0.821,0.853,0.866) | (0.429,0.476,0.5) |
| 4 | (0.349,0.366,0.407) | (0.743,0.871,1) | (0.777,0.889,1) | (0.697,0.774,0.852) | (0.643,0.715,0.75) | (0.898,0.932,0.966) | (0.657,0.676,0.695) | (0.621,0.64,0.647) | (0.429,0.476,0.5) |
| 5 | (0.506,0.532,0.591) | (0.556,0.656,0.759) | (0.408,0.444,0.481) | (0.48,0.533,0.587) | (0.619,0.688,0.722) | (0.932,0.966,1) | (0.779,0.798,0.817) | (0.636,0.655,0.663) | (0.429,0.476,0.5) |
| 6 | (0.371,0.39,0.433) | (0.556,0.656,0.759) | (0.408,0.444,0.481) | (0.654,0.727,0.8) | (0.827,0.918,0.964) | (0.932,0.966,1) | (0.79,0.81,0.829) | (0.608,0.625,0.632) | (0.429,0.476,0.5) |
| 7 | (0.297,0.312,0.346) | (0.556,0.656,0.759) | (0.408,0.444,0.481) | (0.818,0.909,1) | (0.841,0.934,0.98) | (0.915,0.949,0.983) | (0.771,0.79,0.81) | (0.615,0.632,0.64) | (0.857,0.952,1) |
| 8 | (0.675,0.709,0.788) | (0.556,0.656,0.759) | (0.408,0.444,0.481) | (0.36,0.4,0.44) | (0.472,0.524,0.551) | (0.814,0.847,0.881) | (0.619,0.638,0.657) | (0.859,0.894,0.909) | (0.429,0.476,0.5) |
| 9 | (0.768,0.807,0.896) | (0.556,0.656,0.759) | (0.408,0.444,0.481) | (0.316,0.351,0.387) | (0.428,0.476,0.499) | (0.898,0.932,0.966) | (0.676,0.695,0.714) | (0.753,0.78,0.791) | (0.429,0.476,0.5) |
| 10 | (0.495,0.52,0.578) | (0.556,0.656,0.759) | (0.408,0.444,0.481) | (0.491,0.545,0.6) | (0.692,0.768,0.807) | (0.831,0.864,0.898) | (0.752,0.771,0.79) | (0.733,0.759,0.769) | (0.429,0.476,0.5) |
| 11 | (0.857,0.9,1) | (0.527,0.601,0.672) | (0.408,0.444,0.481) | (0.283,0.315,0.347) | (0.472,0.524,0.551) | (0.814,0.847,0.881) | (0.962,0.981,1) | (0.797,0.827,0.84) | (0.214,0.238,0.25) |
| 12 | (0.371,0.39,0.433) | (0.527,0.601,0.672) | (0.408,0.444,0.481) | (0.654,0.727,0.8) | (0.857,0.952,1) | (0.898,0.932,0.966) | (0.867,0.886,0.905) | (0.701,0.724,0.733) | (0.857,0.952,1) |
| 13 | (0.557,0.585,0.65) | (0.527,0.601,0.672) | (0.408,0.444,0.481) | (0.436,0.485,0.533) | (0.681,0.756,0.793) | (0.847,0.881,0.915) | (0.848,0.867,0.886) | (0.743,0.769,0.78) | (0.857,0.952,1) |
| 14 | (0.405,0.425,0.473) | (0.473,0.514,0.556) | (0.408,0.444,0.481) | (0.6,0.667,0.733) | (0.706,0.784,0.824) | (0.814,0.847,0.881) | (0.932,0.951,0.97) | (0.705,0.728,0.738) | (0.429,0.476,0.5) |
| 15 | (0.743,0.78,0.867) | (0.473,0.514,0.556) | (0.408,0.444,0.481) | (0.327,0.363,0.4) | (0.428,0.476,0.499) | (0.898,0.932,0.966) | (0.718,0.737,0.756) | (0.688,0.71,0.719) | (0.429,0.476,0.5) |
| 16 | (0.518,0.544,0.604) | (0.473,0.514,0.556) | (0.408,0.444,0.481) | (0.469,0.521,0.573) | (0.681,0.756,0.793) | (0.847,0.881,0.915) | (0.79,0.81,0.829) | (0.692,0.714,0.724) | (0.429,0.476,0.5) |
| 17 | (0.398,0.418,0.464) | (0.473,0.514,0.556) | (0.408,0.444,0.481) | (0.611,0.679,0.747) | (0.706,0.784,0.824) | (0.814,0.847,0.881) | (0.871,0.89,0.91) | (0.667,0.688,0.696) | (0.429,0.476,0.5) |
| 18 | (0.825,0.867,0.963) | (0.556,0.572,0.585) | (0.408,0.444,0.481) | (0.294,0.327,0.36) | (0.428,0.476,0.499) | (0.898,0.932,0.966) | (0.705,0.724,0.743) | (0.71,0.733,0.743) | (0.214,0.238,0.25) |
| 19 | (0.446,0.468,0.52) | (0.556,0.572,0.585) | (0.408,0.444,0.481) | (0.545,0.606,0.667) | (0.681,0.756,0.793) | (0.847,0.881,0.915) | (0.686,0.705,0.724) | (0.696,0.719,0.728) | (0.429,0.476,0.5) |
| 20 | (0.619,0.65,0.722) | (0.556,0.572,0.585) | (0.408,0.444,0.481) | (0.393,0.436,0.48) | (0.512,0.569,0.597) | (0.746,0.78,0.814) | (0.804,0.823,0.842) | (0.84,0.873,0.887) | (0.429,0.476,0.5) |

**Table A5**
Normalized Fuzzy data II for smartphone measurable features.

| ID | RAM | Max. Ext. Storage | Primary Camera | Front Camera | Max Video Res. | Bat Cap | MaxTalkTime | Release Date |
|---|---|---|---|---|---|---|---|---|
| 1 | (0.286,0.317,0.333) | (0,0,0) | (0.348,0.386,0.386) | (0.083,0.092,0.092) | (0.2,0.25,0.3) | (0.418,0.464,0.487) | (0.36,0.4,0.4) | (0,0.053,0.105) |
| 2 | (0.286,0.317,0.333) | (0,0,0) | (0.348,0.386,0.386) | (0.083,0.092,0.092) | (0.2,0.25,0.3) | (0.418,0.464,0.487) | (0.36,0.4,0.4) | (0,0.053,0.105) |
| 3 | (0.286,0.317,0.333) | (0,0,0) | (0.348,0.386,0.386) | (0.083,0.092,0.092) | (0.2,0.25,0.3) | (0.482,0.535,0.562) | (0.504,0.56,0.56) | (0.632,0.684,0.737) |
| 4 | (0.286,0.317,0.333) | (0,0,0) | (0.348,0.386,0.386) | (0.083,0.092,0.092) | (0.2,0.25,0.3) | (0.776,0.862,0.905) | (0.864,0.96,0.96) | (0.632,0.684,0.737) |
| 5 | (0.857,0.952,1) | (0.25,0.5,0.5) | (0.565,0.628,0.628) | (0.145,0.162,0.162) | (0.9,0.95,1) | (0.852,0.946,0.994) | (0.756,0.84,0.84) | (0,0.053,0.105) |
| 6 | (0.857,0.952,1) | (0.5,1,1) | (0.696,0.773,0.773) | (0.256,0.285,0.285) | (0.9,0.95,1) | (0.857,0.952,1) | (0.72,0.8,0.8) | (0.632,0.684,0.737) |
| 7 | (0.857,0.952,1) | (0.5,1,1) | (0.696,0.773,0.773) | (0.256,0.285,0.285) | (0.9,0.95,1) | (0.799,0.887,0.932) | (0.648,0.72,0.72) | (0.737,0.789,0.842) |
| 8 | (0.571,0.635,0.667) | (0.25,0.5,0.5) | (0.565,0.628,0.628) | (0.346,0.385,0.385) | (0.2,0.25,0.3) | (0.612,0.68,0.714) | (0.612,0.68,0.68) | (0.789,0.842,0.895) |
| 9 | (0.571,0.635,0.667) | (0.25,0.5,0.5) | (0.565,0.628,0.628) | (0.346,0.385,0.385) | (0.2,0.25,0.3) | (0.785,0.873,0.916) | (0.576,0.64,0.64) | (0.895,0.947,1) |
| 10 | (0.571,0.635,0.667) | (0.5,1,1) | (0.696,0.773,0.773) | (0.145,0.162,0.162) | (0.9,0.95,1) | (0.745,0.828,0.87) | (0.756,0.84,0.84) | (0.368,0.421,0.474) |
| 11 | (0.286,0.317,0.333) | (0.125,0.25,0.25) | (0.348,0.386,0.386) | (0.09,0.1,0.1) | (0.2,0.25,0.3) | (0.676,0.751,0.789) | (0.54,0.6,0.6) | (0.579,0.632,0.684) |
| 12 | (0.857,0.952,1) | (0.5,1,1) | (0.565,0.628,0.628) | (0.145,0.162,0.162) | (0.9,0.95,1) | (0.799,0.887,0.932) | (0.756,0.84,0.84) | (0.632,0.684,0.737) |
| 13 | (0.571,0.635,0.667) | (0,0,0) | (0.565,0.628,0.628) | (0.145,0.162,0.162) | (0.2,0.25,0.3) | (0.799,0.887,0.932) | (0.612,0.68,0.68) | (0,0.053,0.105) |
| 14 | (0.571,0.635,0.667) | (0.5,1,1) | (0.565,0.628,0.628) | (0.346,0.385,0.385) | (0.2,0.25,0.3) | (0.719,0.799,0.839) | (0.864,0.96,0.96) | (0.632,0.684,0.737) |
| 15 | (0.571,0.635,0.667) | (0.5,1,1) | (0.565,0.628,0.628) | (0.554,0.615,0.615) | (0.2,0.25,0.3) | (0.692,0.769,0.807) | (0.828,0.92,0.92) | (0.737,0.789,0.842) |
| 16 | (0.571,0.635,0.667) | (0.5,1,1) | (0.565,0.628,0.628) | (0.9,1,1) | (0.2,0.25,0.3) | (0.639,0.71,0.745) | (0.72,0.8,0.8) | (0.737,0.789,0.842) |
| 17 | (0.571,0.635,0.667) | (0.5,1,1) | (0.174,0.193,0.193) | (0.346,0.385,0.385) | (0.2,0.25,0.3) | (0.692,0.769,0.807) | (0.72,0.8,0.8) | (0.368,0.421,0.474) |
| 18 | (0.286,0.317,0.333) | (0.125,0.25,0.25) | (0.348,0.386,0.386) | (0.346,0.385,0.385) | (0.2,0.25,0.3) | (0.665,0.739,0.776) | (0.9,1,1) | (0.579,0.632,0.684) |
| 19 | (0.857,0.952,1) | (0.5,1,1) | (0.9,1,1) | (0.152,0.169,0.169) | (0.9,0.95,1) | (0.825,0.917,0.963) | (0.684,0.76,0.76) | (0.632,0.684,0.737) |
| 20 | (0.571,0.635,0.667) | (0.5,1,1) | (0.9,1,1) | (0.152,0.169,0.169) | (0.9,0.95,1) | (0.692,0.769,0.807) | (0.504,0.56,0.56) | (0.632,0.684,0.737) |

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